Role of Images and Preimages in Topology: Lattice-Valued Case

This talk continues the theme of the role of images and preimages in topology, but extends our previous discussion of 19 March 2012 to the $L$-valued case for so-called $L$-topology, where $L$ is a (semi)frame or quantale or any other appropriate complete lattice with possibly additional structure. Hence this talk focuses on the $L$-powerset monad comprising the $L$-powerset of a set (all mappings from the set to $L$), together with the ($L$-)Zadeh image operator and the ($L$-)Zadeh preimage operator and the $L$-lower image operator of a function and the adjoint relationships of these operators. The Zadeh image and preimage operators were introduced by Zadeh in 1965, and the $L$-lower image operator by Rodabaugh in 2007.

As with traditional topology, this trio of operators, especially the first two, provide $L$-topology with much of its functional structure and important results and applications; and this trio insures that traditional topology is sufficiently emulated by $L$-topology that many ideas of the former can be extended and generalized to the latter. Examples: the image operator gives a point-wise version of $L$-continuity; the preimage operator gives global $L$-continuity; the special properties of the Zadeh preimage operator insure the existence of initial and final structures—the hallmark of a category being “topological”—so that the category $L$-Top is topological just as Top is topological; and the whole powerset monad provides the content of the proofs that $L$-continuity preserves various compactness and connectivity axioms proposed for $L$-topology. The proof of $L$-continuity’s preservation of Chang’s compactness—this compactness precisely imitates the standard compactness axiom for locales—will be examined in detail to see that the $L$-powerset monad provides all of that proof except the definition of Chang’s compactness.

With the foregoing as motivation, we will examine the production of the image and preimage operators and answer the question of why the preimage operator has so many good properties which anchor $L$-topology, $L$-measure theory, and other areas of lattice-valued mathematics; and this examination, analogous to the case with the traditional powerset monad, has a necessary role for relations and their Zadeh image and preimage operators.