Due 09-06-23

(1) 1. Willard: 1A (4880/5880), 1B (1, 2), 1B (3) (6980).

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(1) 2. Willard: 1C (4880/5880), 1D (3) (6980).

3. Prove that the composition of two 1–1 functions is 1–1.

4. Prove that the composition of two onto functions is onto.

Due 09-08-23

5. Let \( X \) be an infinite subset of \( \mathbb{N} \). Prove that \( |X| = |\mathbb{N}| \).

(1) 6.

a. Let \( X \) and \( Y \) be sets. Prove that \( |X| \leq |Y| \) if and only if there exists a function from \( Y \) onto \( X \). (4880/5880)

b. Consider the proof of the lemma used to prove the CSB theorem. Suppose that \( X = \mathbb{N}, Y = 2\mathbb{N}, \) and \( f : X \to Y \) is defined by \( f(n) = 4n \). What are the sets \( Z \) and \( X \setminus Z \) in terms of \( \mathbb{N} \)? For example, note that all odd natural numbers are elements of \( X_1 \setminus Y_1 \) and hence elements of \( Z \). (6980)

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(1) 7.

a. If possible, give an example of a set \( X \) such that \( |\mathcal{P}(X)| = |\mathbb{N}| \). If this is not possible, prove that it is not. (4880/5880)

b. Let \( X \) be a set. Prove that \( |\mathcal{P}(X)| = |2^X| \). (6980)

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(1) 8. Prove that \( \{A \subseteq \mathbb{N} : A \text{ is finite} \} \) is countable.

(1) 9. Prove that a set is infinite if and only if it is equipotent to a proper subset.

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(3) 10. Let \( X \) be a metric space.

\( i \) Prove that \( X \) and \( \emptyset \) are closed.

\( ii \) Prove that the intersection of any collection of closed sets is closed.

\( iii \) Prove that the union of any finite collection of closed sets is closed.
11. Let \((X,d)\) be a metric space. Define \(\rho : X \times X \to \mathbb{R}\) by \(\rho(x,y) = \frac{d(x,y)}{d(x,y) + 1}\).

(1) a. Prove that \(\rho\) is a metric on \(X\).

(1) b. Prove that for any \(x \in X\) and any \(\varepsilon > 0\), \(B_\rho(x,\varepsilon)\) is open in \((X,d)\).

(1) c. Prove that for any \(x \in X\) and any \(\varepsilon > 0\), \(B_d(x,\varepsilon)\) is open in \((X,\rho)\).