Homework 5880/6980

Due 09-06-23

(1) 1. Willard: **1A** (4880/5880), 1B (1, 2), 1B (**3**) (6980).

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(1) 2. Willard: **1C** (4880/5880), **1D (3)** (6980).

3. Prove that the composition of two 1–1 functions is 1–1.

4. Prove that the composition of two onto functions is onto.

Due 09-08-23

5. Let X be an infinite subset of $\mathbb{N}$. Prove that $|X| = |\mathbb{N}|$.

(1) 6.

a. Let X and Y be sets. Prove that $|X| \leq |Y|$ if and only if there exists a function from Y onto X. (4880/5880)

b. Consider the proof of the lemma used to prove the CSB theorem. Suppose that $X = \mathbb{N}$, $Y = 2\mathbb{N}$, and $f : X \to Y$ is defined by $f(n) = 4n$. What are the sets Z and $X \setminus Z$ in terms of $\mathbb{N}$? For example, note that all odd natural numbers are elements of $X_1 \setminus Y_1$ and hence elements of Z. (6980)

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(1) 7.

a. If possible, give an example of a set X such that $|\mathcal{P}(X)| = |\mathbb{N}|$. If this is not possible, prove that it is not. (4880/5880)

b. Let X be a set. Prove that $|\mathcal{P}(X)| = |2^X|$. (6980)

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(1) 8. Prove that $\{A \subseteq \mathbb{N} : A \text{ is finite}\}$ is countable.

(1) 9. Prove that a set is infinite if and only if it is equipotent to a proper subset.

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(3) 10. Let X be a metric space.

(i) Prove that X and $\emptyset$ are closed.

(ii) Prove that the intersection of any collection of closed sets is closed.

(iii) Prove that the union of any finite collection of closed sets is closed.

11. Let (X,d) be a metric space. Define $\rho : X \times X \to \mathbb{R}$ by $\rho(x,y) = \dfrac{d(x,y)}{d(x,y)+1}$.

(1) a. Prove that ρ is a metric on X.

(1) b. Prove that for any $x \in X$ and any $\varepsilon > 0$, $B_\rho(x,\varepsilon)$ is open in (X,d).

(1) c. Prove that for any $x \in X$ and any $\varepsilon > 0$, $B_d(x,\varepsilon)$ is open in (X,ρ).

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(1) 12. Let (X,d) and (Y,ρ) be metric spaces and suppose that $f : X \to Y$ is an isometry. Prove the following.

(i) The function f is 1–1.

(ii) The function f is continuous.

(iii) If f is onto, then f^{-1} is continuous.

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13. (1) a. Suppose that a Cauchy sequence in a metric space has a convergent subsequence. Prove that the sequence converges.

(1) b. Suppose (X,d) and (Y,ρ) are metric spaces, $f : X \to Y$, and $x \in X$. Prove that the following two statements are equivalent.

(i) The function f is continuous at x.

(ii) If $\{x_n\}_{n=1}^{\infty}$ is any sequence in X that converges to x, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$.

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(1) 14.

a. Willard 3F (1) (4880/5880).

b. Willard 3F (2) (6980).

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15. Willard 3D.

(1) 16. Willard 3D (3).

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17. Willard 3H.

(1) 18. Willard 3H (3).

Due 10-04-22

(1) 19.

a. Prove that $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$ with the usual topology. (5880)

b. Prove that $\mathbb{Q}$ is not a G_δ-set in the Michael Line. Conclude that the Michael Line is not metrizable (Willard 3H: 3). (6980)

Due 10-09-23

20. Prove that the composition of continuous functions is continuous.

(1) 21. a. Prove that metrizability is a topological property. (4880/5880)

b. Prove that $\mathbb{R}$ is homeomorphic to (-1,1). (6980)

Due 10-13-23

22. For each $a,d \in \mathbb{N}$, set $B(a,d) = \{a + nd : n \in \omega\} = \{a, a+d, a+2d, a+3d, a+4d, \ldots\}$.
Set $\mathcal{B} = \{B(a,d) : a,d \in \mathbb{N}\}$.

(1) a. Prove that $\mathcal{B}$ is a base for a topology on $\mathbb{N}$.

Set $\mathcal{T}$ equal to the topology on $\mathbb{N}$ generated by $\mathcal{B}$.

b. Prove that $\{1\}$ is not open in $(\mathbb{N},\mathcal{T})$.

Set P equal to the set of positive prime numbers.

(1) c. Prove that $\{1\} = \bigcap_{p \in P} B(1,p)$.

d. Conclude that P is infinite.

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(1) 23. Willard: 5D.

Due 10-16-23

24. Let X be a topological space and suppose that $Y \subseteq X$. Prove that $E \subseteq Y$ is closed in Y if and only if there exists a closed in X set F such that $E = Y \cap F$.

25. Let Y be a subspace of a space X and suppose that $A \subseteq Y$. Prove that $cl_Y(A) = cl_X(A) \cap Y$.

Due 10-18-23

(1) 26.
a. Suppose that X and Y are topological spaces, Y is Hausdorff, and $f : X \to Y$ is 1–1 and continuous. Prove that X is Hausdorff (4880/5880).

b. Suppose that X and Y are topological spaces, Y is Hausdorff, D is dense in X, $f : X \to Y$ and $g : X \to Y$ are continuous, and $f(x) = g(x)$ for all $x \in D$. Prove that $f(x) = g(x)$ for all $x \in X$ (6980).

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(1) 27. Willard: 14 D.

Due 10-25-23

28. Willard: 15 B 2.

(1) 29. Let X be a topological space. Prove the following are equivalent:

(i) X is completely normal;

(ii) every subspace of X is normal;

(iii) every open subspace of X is normal.

Due 11-01-23

(1) 30.

a. Prove that a closed subset of a compact set is compact (4880/5880).

b. Prove that a compact subset of a Hausdorff space is closed (6980).

31. Suppose that $(X,\mathcal{T})$ is a compact Hausdorff topological space. Prove the following.

(1) a. If σ is a topology on X that is strictly stronger than $\mathcal{T}$, then (X,σ) is not compact.

(1) b. If σ is a topology on X that is strictly weaker than $\mathcal{T}$, then (X,σ) is not Hausdorff.

Due 11-15-23

(1) 32. Willard 17 F 6.

(1) 33. Prove that a countably compact metric space is compact.

Due 11-15-23

(1) 34. Let X be a Lindelöf space. Prove that every uncountable subset of X has a condensation point.

Due 11-15-23

(1) 35. a. Willard 16 A: 3. (4880/5880)

b. Prove that a compact Hausdorff space in which every singleton is a a G_δ-set is first countable. (6980)

(1) 36. Willard 16 B: 2.

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(1) 37. For any metric space (X,d), TFAE:

a. (X,d) is second countable;

b. (X,d) is Lindelöf;

c. (X,d) is separable.

(1) 38. Prove that every collection of disjoint open sets in a separable space must be countable.

39. Willard 16 B: 3.

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(1) 40. Prove the Tube Lemma.

(1) 41. Prove that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

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(3) 42. Willard 8 D: 1–3.

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(1) 43. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $a < b$. Prove that $f([a,b])$ is a closed interval or a singleton.

(1) 44. Prove that a countable Tychonoff space has a base of clopen sets.

(1) 45. Willard 26 A: 1, **3**, 4.