Contents

Chapter 1: Summer 2004

Section 1.1: Exam 1

Exam 1 Math 2673 Summer 2004

Name:__________________________

E-mail (optional):__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

1. Let $\boldsymbol{u} = \langle 2, 1, \text{-}2 \rangle$, $\boldsymbol{v} = \langle 1, 2, \text{-}1 \rangle$, and $\boldsymbol{w} = \langle \text{-}3, 0, 1 \rangle$. Also, let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(3) a. Find $\cos\theta$.

(4) b. Find $\sin\theta$.

(4) c. Find the area of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$.

(4) d. Find the volume of the parallelepiped determined by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

(3) e. How many vectors in the xy-plane are orthogonal to $\boldsymbol{v}$?

(6) f. Find two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with the following three properties:

i. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{w}$;

ii. $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel;

iii. $\boldsymbol{b}$ and $\boldsymbol{v}$ are orthogonal.

2. Let $\boldsymbol{f}(t) = \langle t^3, \sin t \rangle$.

(4) a. Sketch the graph of $\boldsymbol{f}$ in the direction of increasing t.

(4) b. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that the graph of g coincides with the graph of $\boldsymbol{f}$.

3. Let $\boldsymbol{f}(t) = \langle t^2, e^t, \sin t \rangle$.

(3) a. Find $\boldsymbol{f}'(t)$.

(4) b. Find the equation of the line that is tangent to the graph of $\boldsymbol{f}$ where $t = \frac{\pi}{3}$.

(3) c. Calculate $\int_0^1 \boldsymbol{f}(t)\, dt$.

4. A projectile is launched due north at an angle of $\frac{\pi}{4}$ to the horizontal at an initial speed of 100 ft/sec. The wind is blowing northeast at a rate of 50 ft/sec.

(3) a. Find the initial velocity vector.

(6) b. Find the vector functions that describe velocity and motion.

(3) c. Find the maximum height.

(3) d. Find the horizontal range.

(3) e. Find the speed of impact.

(4) 5. Give the equation of the plane that contains the points (1,-1,1), (3,0,1), and (-2,1,0).

(4) 6. Do the planes with the equations $2x + y - z = 1$ and -$x + 3y + 4z = 3$ intersect? If so, where?

7. Consider the rectangular equation $x^2 + y^2 + z^2 + 2x - 4y = 4$.

(3) a. Describe the object defined by the equation.

(3) b. Write the equation using cylindrical coordinates.

(3) c. Write the equation using spherical coordinates.

8. Calculate the following limits.

(4) a. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 + 2xy + 4y}{x^2y + 1}$

(4) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

(6) 9. Use the $\varepsilon - \delta$ definition of limit to prove that $\displaystyle\lim_{(x,y)\to(1,1)} \big(2x + y\big) = 3.$

10. Let $f(x,y) = x^2y - 2x - y$.

(6) a. Find all local extrema and saddle points.

(6) b. Find the absolute extrema of f on the region bounded by the triangle with vertices (0,0), (3,0), and (0,-3).

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11. Let $\boldsymbol{u} = \langle 2, 1, \text{-}2 \rangle$, $\boldsymbol{v} = \langle 1, 2, \text{-}1 \rangle$, and $\boldsymbol{w} = \langle \text{-}3, 0, 1 \rangle$. Also, let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(3) a. Find $\cos\theta$.

$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{\text{-}4}{\sqrt{60}} = \frac{\text{-}2}{\sqrt{15}}$

(4) b. Find $\sin\theta$.

$\sin\theta = \frac{\|\boldsymbol{v}\times\boldsymbol{w}\|}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{\|\langle 2,2,6\rangle\|}{\sqrt{60}} = \frac{\sqrt{44}}{\sqrt{60}} = \frac{\sqrt{11}}{\sqrt{15}}$

(4) c. Find the area of the parallelogram determined by $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\|\boldsymbol{v} \times \boldsymbol{w}\| = \| \langle 2,2,6 \rangle \| = \sqrt{44}$

(4) d. Find the volume of the parallelepiped determined by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

$$\left\| \begin{array}{ccc} 2 & 1 & \text{-}2 \\ 1 & 2 & \text{-}1 \\ \text{-}3 & 0 & 1 \end{array} \right\| = |2(2-0) - (1-3) - 2(0+6)| = 6$$

(3) e. How many vectors are parallel to the xy-plane and orthogonal to $\boldsymbol{v}$?

A vector $\boldsymbol{x} = \langle x,y,0\rangle$ which is parallel to the xy-plane is orthogonal to $\boldsymbol{v}$ if and only if $\boldsymbol{v}\cdot\boldsymbol{x} = x + 2y = 0$. Therefore, there are infinitely many vectors that are both parallel to the xy-plane and orthogonal to $\boldsymbol{v}$.

(6) f. Find two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with the following three properties:

i. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{w}$;

ii. $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel;

iii. $\boldsymbol{b}$ and $\boldsymbol{v}$ are orthogonal.

Let $\boldsymbol{a} = \text{proj}_{\boldsymbol{w}}\boldsymbol{v} = \left\langle \text{-}\frac{2}{3}, \text{-}\frac{4}{3}, \frac{2}{3} \right\rangle$ and $\boldsymbol{b} = \boldsymbol{w} - \boldsymbol{a} = \boldsymbol{w} - \text{proj}_{\boldsymbol{w}}\boldsymbol{v} = \left\langle \text{-}\frac{7}{3}, \frac{4}{3}, \frac{1}{3} \right\rangle$.

12. Let $\boldsymbol{f}(t) = \langle t^3, \sin t \rangle$.

(4) a. Sketch the graph of $\boldsymbol{f}$ in the direction of increasing t.

(4) b. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that the graph of g coincides with the graph of $\boldsymbol{f}$.

$g(x) = \sin\sqrt[3]{x}$

13. Let $\boldsymbol{f}(t) = \langle t^2, e^t, \sin t\rangle$.

(3) a. Find $\boldsymbol{f}'(t)$.

$\boldsymbol{f}'(t) = \langle 2t, e^t, \cos t\rangle$.

(4) b. Find the equation of the line that is tangent to the graph of $\boldsymbol{f}$ where $t = \frac{\pi}{3}$.

$\boldsymbol{f}\left(\frac{\pi}{3}\right) = \left\langle \frac{\pi^2}{9}, e^{\frac{\pi}{3}}, \frac{\sqrt{3}}{2}\right\rangle$

$\boldsymbol{f}'\left(\frac{\pi}{3}\right) = \left\langle \frac{2\pi}{3}, e^{\frac{\pi}{3}}, \frac{1}{2}\right\rangle$

$x = \frac{2\pi}{3}t + \frac{\pi^2}{9}$

$y = e^{\frac{\pi}{3}}t + e^{\frac{\pi}{3}}$

$z = \frac{1}{2}t + \frac{\sqrt{3}}{2}$

(3) c. Calculate $\displaystyle\int_0^1 \boldsymbol{f}(t)\,dt$.

$$\int_0^1 \boldsymbol{f}(t)\,dt = \left\langle \int_0^1 t^2\,dt, \int_0^1 e^t\,dt, \int_0^1 \sin t\,dt \right\rangle = \left\langle \frac{1}{3}t^3\Big|_0^1, e^t\Big|_0^1, -\cos t\Big|_0^1 \right\rangle = \left\langle \frac{1}{3}, e-1, 1-\cos 1\right\rangle$$

14. A projectile is launched due north at an angle of $\frac{\pi}{4}$ to the horizontal at an initial speed of 10 ft/sec. The wind is blowing northeast at a rate of 5 ft/sec.

Note 1: The 10 and 5 above were originally 100 and 50. The solution, however, is written for 5 and 10. Rather than change the solution, I changed the problem. Please see me if you have any questions.

(3) a. Find the initial velocity vector.

$\boldsymbol{v}(0) = \langle 0, -5\sqrt{2}, 5\sqrt{2}\rangle$

(6) b. Find the vector functions that describe velocity and motion.

$\boldsymbol{a}(0) = \langle 5, -5, -32\rangle$

$\boldsymbol{v}(t) = \langle 5t, -5t, -32t\rangle + \boldsymbol{v}(0)$

$= \langle 5t, -5t - 5\sqrt{2}, -32t + 5\sqrt{2}\rangle$

$\boldsymbol{s}(t) = \left\langle \frac{5}{2}t^2, -\frac{5}{2}t^2 - 5\sqrt{2}t, -16t^2 + 5\sqrt{2}t\right\rangle + \boldsymbol{s}(0)$

$\boldsymbol{s}(t) = \left\langle \frac{5}{2}t^2, -\frac{5}{2}t^2 - 5\sqrt{2}t, -16t^2 + 5\sqrt{2}t\right\rangle$

(3) c. Find the maximum height.

$-32t + 5\sqrt{2} = 0$

$t = \frac{5\sqrt{2}}{32}$

$-16\left(\frac{5\sqrt{2}}{32}\right)^2 + 5\sqrt{2}\left(\frac{5\sqrt{2}}{32}\right)$

$= -\frac{25}{32} + \frac{50}{32}$

$= \frac{25}{32}$

(3) d. Find the position of impact.

$-16t^2 + 5\sqrt{2}t = 0$

$\frac{5\sqrt{2}}{16}$

$$\boldsymbol{s}\left(\frac{5\sqrt{2}}{16}\right) = \left\langle \frac{5}{2}\left(\frac{5\sqrt{2}}{16}\right)^2, -\frac{5}{2}\left(\frac{5\sqrt{2}}{16}\right)^2 - 5\sqrt{2}\left(\frac{5\sqrt{2}}{16}\right), -16\left(\frac{5\sqrt{2}}{16}\right)^2 + 5\sqrt{2}\left(\frac{5\sqrt{2}}{16}\right)\right\rangle = \left\langle \frac{125}{256}, -\frac{925}{256}, 0\right\rangle$$

(3) e. Find the speed of impact.

$\left\|\boldsymbol{v}\left(\frac{5\sqrt{2}}{16}\right)\right\| = \frac{95}{8}$

(4) 15. Give the equation of the plane that contains the points (1,-1,1), (3,0,1), and (-2,1,0).

$\boldsymbol{v} = \langle 3-1, 0+1, 1-1\rangle = \langle 2,1,0\rangle$

$\boldsymbol{w} = \langle 3+2, 0-1, 1-0\rangle = \langle 5,-1,1\rangle$

$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle 1,-2,-7\rangle$

$1(x-1) - 2(y+1) - 7(z-1) = 0$

(4) 16. Do the planes with the equations $2x + y - z = 1$ and $-x + 3y + 4z = 3$ intersect? If so, find the line of intersection?

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ -x & + & 3y & = & 3 \end{array}$$

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ -2x & + & 6y & = & 6 \end{array}$$

$7y = 7$

$y = 1$

$x = 0$

$z = 0$

$(0, 1, 0)$

$$\begin{array}{rcrcl} 2x & - & z & = & 1 \\ -x & + & 4z & = & 3 \end{array}$$

$$\begin{array}{rcrcl} 2x & - & z & = & 1 \\ -2x & + & 8z & = & 6 \end{array}$$

$7z = 7$

$z = 1$

$x = 1$

$y = 0$

$(1, 0, 1)$

$\boldsymbol{v} = \langle 1,-1,1\rangle$

The line of intersection is given by

$x = t$

$y = -t + 1$

$z = t.$

17. Consider the rectangular equation $x^2 + y^2 + z^2 + 2x - 4y = 4$.

(3) a. Describe the object defined by the equation.

$x^2 + y^2 + z^2 + 2x - 4y = 4$

$(x+1)^2 + (y-2)^2 + z^2 = 9$

This is the equation of the sphere with center (-1,2,0) and radius 3.

(3) b. Write the equation using cylindrical coordinates.

$x^2 + y^2 + z^2 + 2x - 4y = 4$

$r^2 \cos^2\theta + r^2 \sin^2\theta + z^2 + 2r\cos\theta - 4r\sin\theta = 4$

$r^2 + z^2 + 2r\cos\theta - 4r\sin\theta = 4$

(3) c. Write the equation using spherical coordinates.

$x^2 + y^2 + z^2 + 2x - 4y = 4$

$\rho^2 + 2\rho\sin\phi\cos\theta - 4\rho\sin\phi\sin\theta = 4$

18. Calculate the following limits.

(4) a. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 + 2xy + 4y}{x^2y + 1} = \tfrac{7}{2}$

(4) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$

$\displaystyle\lim_{(x,0)\to(0,0)} \frac{xy}{x^2 + y^2} = 0$

$\displaystyle\lim_{(x,x)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \tfrac{1}{2}$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ DNE

(6) 19. Use the $\varepsilon - \delta$ definition of limit to prove that

$\displaystyle\lim_{(x,y)\to(1,1)} \big(2x + y\big) = 3.$

Proof: Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{3}$. Suppose that $d((x,y),(1,1)) < \delta$. Then

$\sqrt{(x-1)^2 + (y-1)^2} < \delta$

which implies

$\sqrt{(x-1)^2} = |x-1| < \delta$

and

$\sqrt{(y-1)^2} = |y-1| < \delta.$

So

$2|x-1| + |y-1| < 3\delta$

$2|x-1| + |y-1| < \varepsilon$

$|2(x-1) + y - 1| < \varepsilon$

$|2x - 2 + y - 1| < \varepsilon$

$|2x + y - 3| < \varepsilon$

as desired. ■

20. Let $f(x,y) = x^2y - 2x - y$.

(6) a. Find all local extrema and saddle points.

$f_x(x,y) = 2xy - 2$

$f_y(x,y) = x^2 - 1$

$f_{xy}(x,y) = 2x$

$f_{xx}(x,y) = 2y$

$f_{yy}(x,y) = 0$

CP: (1,1), (-1,-1)

$D(1,1) =$ -4

$D(-1,-1) =$ -4

SP: (1,1,-2), (-1,-1,2)

(6) b. Find the absolute extrema of f on the region bounded by the triangle with vertices (0,0), (3,0), and (0,-3).

Note that neither critical point is in or on the triangle so we analyze the function on the boundary.

Case 1: $0 \leq x \leq 3$ and $y = 0$

$f(x,0) =$ -$2x$

Max: 0 at (0,0)

Min: -6 at (3,0)

Case 2: $x = 0$ and $-3 \leq y \leq 0$

$f(0,y) =$ -y

Max: 3 at (0,-3)

Min: 0 at (0,0)

Case 3: $0 \leq x \leq 3$ and $y = x - 3$

$f(x,x-3)$

$= x^2(x-3) - 2x - (x-3)$

$= x^3 - 3x^2 - 3x + 3$

$g(x) = x^3 - 3x^2 - 3x + 3$

$g'(x) = 3x^2 - 6x - 3$

$g'(x) = 3(x^2 - 2x - 1)$

$3(x^2 - 2x - 1) = 0$

$x = \frac{2 \pm \sqrt{8}}{2}$

$x = 1 \pm \sqrt{2}$

CP: $1 + \sqrt{2}$

$g(0) = 3$

$g(3) =$ -6

$g\left(1 + \sqrt{2}\right) = -2 - 4\sqrt{2}$

Max: 3 at (0,-3)

Min: $-2 - 4\sqrt{2}$ at $\left(1 + \sqrt{2}, -2 + \sqrt{2}\right)$

Section 1.2: Exam 2

Exam 2 Math 2673 Spring 2004

Name:__________________________

E-mail (optional):________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

21. Define $\boldsymbol{f} : \mathbb{R} \to \boldsymbol{V}_3$, $\boldsymbol{g} : \mathbb{R} \to \boldsymbol{V}_3$, and $h : \mathbb{R} \to \mathbb{R}$ as follows:

$\boldsymbol{f}(t) = \langle t^2, e^t, t \rangle$ $\qquad$ $\boldsymbol{g}(t) = \left\langle \sqrt{t}, \ln t; , t \right\rangle$ $\qquad$ $h(t) = t^3$

a. For each of the following, give the rule for the given function and find the domain (if possible).

(2) ***i.*** $(h\boldsymbol{f})(t)$

(2) ***ii.*** $\boldsymbol{g}[h(t)]$

(2) ***iii.*** $(\boldsymbol{f} + \boldsymbol{g})(t)$

(2) ***iv.*** $\boldsymbol{g}[\boldsymbol{f}(t)]$

b. Differentiate.

(2) ***i.*** $\boldsymbol{f}(t)$

(2) ***ii.*** $\boldsymbol{g}(t)$

c. Integrate.

(2) ***i.*** $\displaystyle\int_0^1 \boldsymbol{f}(t)\, dt$

Definition 2: Suppose that $\mathcal{C}$ is a plane curve defined by a vector function $\boldsymbol{r}$. Then the **binormal vector** at the point defined by t is $\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$.

22. Let $\mathcal{C}$ be the curve defined by $\boldsymbol{r}(t) = \left\langle 2\sin t, \sqrt{5}t, 2\cos t \right\rangle$.

(3) a. Find the unit tangent vector $\boldsymbol{T}(t)$.

(3) b. Find the unit normal vector $\boldsymbol{N}(t)$.

(3) c. Find the binormal vector $\boldsymbol{B}(t)$.

(3) d. Find the curvature $\kappa(t)$.

23. A projectile is launched with an initial speed of 100 feet per second at an angle of $\frac{\pi}{6}$ to the horizontal. Assume that the only force acting on the object is gravity.

(2) a. Find the initial velocity vector.

(2) b. Find the vector function that describes velocity.

(2) c. Find the vector function that describes motion.

(2) d. Find the maximum height.

(2) e. Find the horizontal range.

(2) f. Find the speed of impact.

(4) 24. Let $\mathcal{C}$ be the curve defined by the vector function $\boldsymbol{r}(t) = \left\langle \cos t, \sin t, \sqrt{3}t \right\rangle$. Paramaterize the curve with respect to arc length.

25. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = \sqrt{x^2 + y^2 - z}$.

(2) a. Give the domain of f.

(2) b. Calculate $f(2,1,1)$.

26. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = 4x^2 + 9y^2$.

a. Sketch the c-level curve for each of the following values of c.

(2) ***i.*** $c = 1$

(2) ***ii.*** $c = 36$

(3) b. Sketch the graph of f.

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27. Define $\boldsymbol{f} : \mathbb{R} \to \boldsymbol{V}_3$, $\boldsymbol{g} : \mathbb{R} \to \boldsymbol{V}_3$, and $h : \mathbb{R} \to \mathbb{R}$ as follows:

$\boldsymbol{f}(t) = \langle t^2, e^t, t \rangle$ $\quad\quad$ $\boldsymbol{g}(t) = \left\langle \sqrt{t}, \ln t, t \right\rangle$ $\quad\quad$ $h(t) = t^3$

a. For each of the following, give the rule for the given function and find the domain (if possible).

(2) ***i.*** $(h\boldsymbol{f})(t) = \langle t^5, t^3 e^t, t^4 \rangle$

Domain: $\mathbb{R}$

(2) ***ii.*** $\boldsymbol{g}[h(t)] = \left\langle \sqrt{t^3}, \ln t^3, t^3 \right\rangle$

Domain: $(0,\infty)$

(2) ***iii.*** $(\boldsymbol{f} + \boldsymbol{g})(t) = \left\langle t^2 + \sqrt{t}, e^t + \ln t, 2t \right\rangle$

Domain: $(0,\infty)$

(2) ***iv.*** $\boldsymbol{g}[\boldsymbol{f}(t)]$

This is undefined since the range of $\boldsymbol{f}$ consists of vectors and the domain of $\boldsymbol{g}$ consists of real numbers.

b. Differentiate.

(2) ***i.*** $\boldsymbol{f}(t) = \langle t^2, e^t, t \rangle$

$\boldsymbol{f}'(t) = \langle 2t, e^t, 1 \rangle$

(2) ***ii.*** $\boldsymbol{g}(t) = \left\langle \sqrt{t}, \ln t, t \right\rangle$

$\boldsymbol{g}'(t) = \left\langle \frac{1}{2\sqrt{t}}, \frac{1}{t}, 1 \right\rangle$

c. Integrate.

(2) ***i.*** $\displaystyle\int_0^1 \boldsymbol{f}(t)\, dt$

$\displaystyle\int_0^1 \langle t^2, e^t, t \rangle \; ; dt$

$$= \left\langle \int_0^1 t^2\,dt, \int_0^1 e^t\,dt, \int_0^1 t\,dt \right\rangle$$

$$= \left\langle \tfrac{1}{3}t^3 \Big|_0^1, e^t \Big|_0^1, \tfrac{1}{2}t^2 \Big|_0^1 \right\rangle$$

$$= \left\langle \tfrac{1}{3}, e-1, \tfrac{1}{2} \right\rangle$$

(4) 28. Let $\mathcal{C}$ be the curve defined by the vector function $\boldsymbol{f}(t) = \left\langle \cos t, \sin t, \sqrt{3}t \right\rangle$. Paramaterize the curve with respect to arc length.

$\boldsymbol{f}'(t) = \langle \text{-} \sin t, \cos t, \sqrt{3} \rangle.$

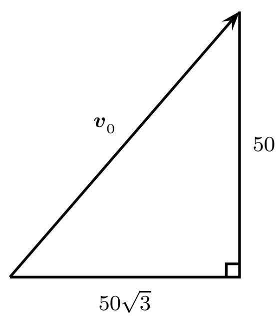
$$s = \int_0^t \sqrt{(\text{-}\sin\tau)^2 + (\cos\tau)^2 + 3}\,d\tau = \int_0^t 2\,d\tau = 2\tau\Big|_0^t = 2t$$

$t = \frac{1}{2}s$

$\boldsymbol{g}(s) = \left\langle \text{-}\sin\left(\frac{1}{2}s\right), \cos\left(\frac{1}{2}s\right), \frac{\sqrt{3}}{2}s \right\rangle.$

29. A projectile is launched with an initial speed of 100 feet per second at an angle of $\frac{\pi}{6}$ to the horizontal. Assume that the only force acting on the object is gravity.

(2) a. Find the initial velocity vector.



$\boldsymbol{v}_0 = 50\sqrt{3}\boldsymbol{i} + 50\boldsymbol{j}$

(2) b. Find the vector function that describes velocity.

$\boldsymbol{a}(t) = \text{-}32\boldsymbol{j}$

$\boldsymbol{v}(t) = \text{-}32t\boldsymbol{j} + \boldsymbol{v}(0)$

$\boldsymbol{v}(t) = \text{-}32t\boldsymbol{j} + 50\sqrt{3}\boldsymbol{i} + 50\boldsymbol{j}$

$\boldsymbol{v}(t) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32t + 50)\boldsymbol{j}$

(2) c. Find the vector function that describes motion.

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j} + \boldsymbol{s}(0)$

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j}$

(2) d. Find the maximum height.

$\boldsymbol{v}(t) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32t + 50)\boldsymbol{j}$

$\text{-}32t + 50 = 0$

$t = \frac{25}{16}$

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j}$

$\text{-}16\left(\frac{25}{16}\right)^2 + 50\left(\frac{25}{16}\right) = \frac{625}{16}$

$\frac{625}{16}$ ft

(2) e. Find the horizontal range.

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j}$

$\text{-}16t^2 + 50t = 0$

$t(\text{-}16t + 50 = 0$

$t = \frac{25}{8}$

$50\sqrt{3} \cdot \frac{25}{8} = \frac{625\sqrt{3}}{4}$

(2) f. Find the speed of impact.

$\left\| \boldsymbol{v}\left(\frac{25}{8}\right) \right\| = 100$

$\boldsymbol{v}(t) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32t + 50)\boldsymbol{j}$

100 ft/sec.

$\boldsymbol{v}\left(\frac{25}{8}\right) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32\left(\frac{25}{8}\right) + 50)\boldsymbol{j}$

30. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = \sqrt{x^2 + y^2 - z}$.

(2) a. Give the domain of f.

$\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 > z\}$

(2) b. Calculate $f(2,1,1)$.

$f(2,1,1) = 2$

31. Let $\mathcal{C}$ be the curve defined by $\boldsymbol{r}(t) = \left\langle 2\sin t, \sqrt{5}t, 2\cos t \right\rangle$.

(3) a. Find the unit tangent vector $\boldsymbol{T}(t)$.

$\boldsymbol{r}'(t) = \left\langle 2\cos t, \sqrt{5}, \text{-}2\sin t \right\rangle$.

$\|\boldsymbol{r}'(t)\| = \sqrt{4\cos^2 t{,}5{,}4\sin^2 t} = 3$

$\boldsymbol{T}(t)$

$= \dfrac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|}$

$= \frac{1}{3}\left\langle 2\cos t{,}\sqrt{5}{,}\text{-}2\sin t \right\rangle$

(3) b. Find the unit normal vector $\boldsymbol{N}(t)$.

$\boldsymbol{T}'(t) = \frac{1}{3}\langle \text{-}2\sin t, 0, \text{-}2\cos t\rangle$

$\|\boldsymbol{T}'(t)\| = \frac{2}{3}$

$\boldsymbol{N}(t) = \frac{3}{2}\boldsymbol{T}'(t) = \langle \text{-}\sin t, 0, \text{-}\cos t\rangle$

(3) c. Find the binormal vector $\boldsymbol{B}(t)$.

$\boldsymbol{B}(t)$

$= \boldsymbol{T}(t) \times \boldsymbol{N}(t)$

$= \frac{1}{3}\left\langle \text{-}\sqrt{5}\cos t, 2, \sqrt{5}\sin t \right\rangle$

(3) d. Find the curvature $\kappa(t)$.

$\kappa(t) = \frac{\|\boldsymbol{T}'(t)\|}{\|\boldsymbol{r}'(t)\|} = \dfrac{\frac{2}{3}}{3} = \frac{2}{9}$

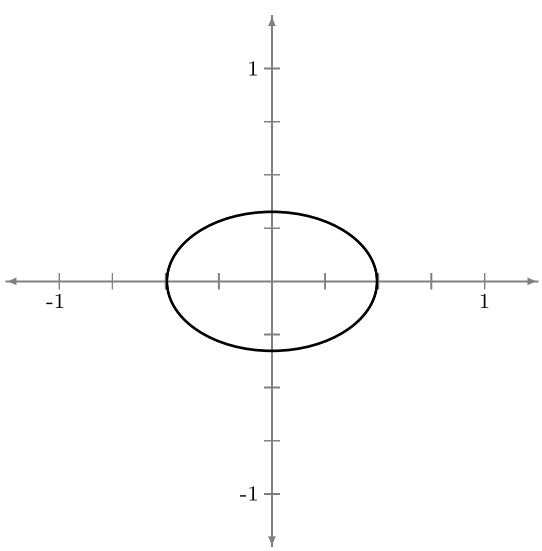
32. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x{,}y) = 4x^2 + 9y^2$.

a. Sketch the c-level curve for each of the following values of c.

(2) ***i.*** $c = 1$

$4x^2 + 9y^2 = 1$

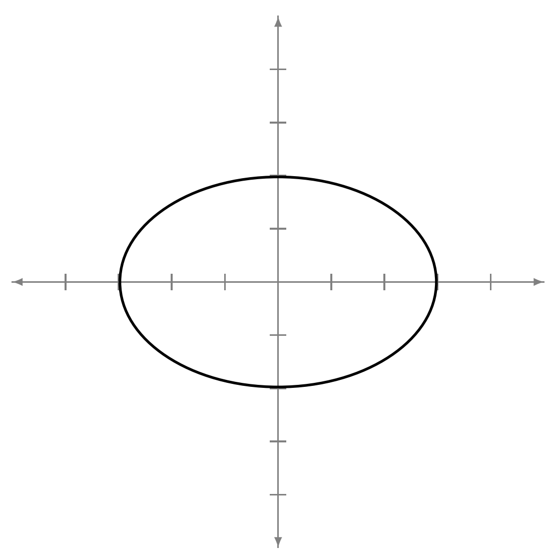
$$\frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{\left(\frac{1}{3}\right)^2} = 1$$



(2) ***ii.*** $c = 36$

$4x^2 + 9y^2 = 36$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$



(3) b. Sketch the graph of f.

Total Points: 312

Section 1.3: Exam 3

Exam 3 Math 2673 Spring 2004

Name:__________________________

E-mail (optional):__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

33. For each of the following, give the coordinates of the point in the the other two coordinate systems.

(2) a. Rectangular coordinates: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{3}\right)$

(2) b. Cylindrical coordinates: $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}, \frac{1}{2}\right)$

(2) c. Spherical coordinates: $\left(3, \frac{\pi}{3}, \frac{\pi}{2}\right)$

34. For each of the following, give the equation of the described object in rectangular, cylindrical, and spherical coordinates.

(6) a. The sphere centered at (0,0,0) with radius 1.

(6) b. The cylinder whose intersection with the xy-plane is the circle centered at (0,0) with radius 3.

35. For each of the following, calculate the limit or show that it does not exist.

(4) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y+3}{x+y+1}$

(4) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

36. Let $f(x,y) = x^2y - y$.

(5) a. Find all local extrema and saddle points.

(5) b. Find absolute extrema of f on the region enclosed by the triangle with vertices (-1,0), (0,0), and (0,1).

(6) 37. Use the definition of limit to show that

$$\lim_{(x,y)\to(1,\text{-}2)} \big(2x + y + 1\big) = 1.$$

(8) 38. Prove that $f(x,y) = 2x + y^2 + 1$ is differentiable for all $(a,b) \in \mathbb{R}^2$ and find $f'_{(a,b)}(x,y)$.

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39. For each of the following, give the coordinates of the point in the the other two coordinate systems.

(2) a.

Rectangular coordinates: $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\sqrt{3}\right)$

$r^2 = x^2 + y^2 = \frac{1}{2} + \frac{1}{2} = 1$

$r = 1$

$\tan\theta = \frac{x}{y} = 1$

$\theta = \frac{\pi}{4}$

$\rho^2 = r^2 + z^2 = 1 + 3 = 4$

$\rho = 2$

$r = \rho\sin\phi$

$1 = 2\sin\phi$

$\sin\phi = \frac{1}{2}$

$\phi = \frac{\pi}{6}$

Cylindrical: $\left(1,\frac{\pi}{4},\sqrt{3}\right)$

Spherical: $\left(2,\frac{\pi}{4},\frac{\pi}{6}\right)$

(2) b.

Cylindrical coordinates: $\left(\frac{\sqrt{3}}{2},\frac{\pi}{6},\frac{1}{2}\right)$

$x = r\cos\theta = \frac{\sqrt{3}}{2}\cos\frac{\pi}{6} = \frac{3}{4}$

$y = r\sin\theta = \frac{\sqrt{3}}{2}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{4}$

$\rho^2 = r^2 + z^2 = \frac{3}{4} + \frac{1}{4} = 1$

$\rho = 1$

$r = \rho\sin\phi$

$\frac{\sqrt{3}}{2} = \sin\phi$

$\phi = \frac{\pi}{3}$

Rectangular: $\left(\frac{3}{4},\frac{\sqrt{3}}{4},\frac{1}{2}\right)$

Spherical: $\left(1,\frac{\pi}{6},\frac{\pi}{3}\right)$

(2) c. Spherical coordinates:

$\left(3,\frac{\pi}{3},\frac{\pi}{2}\right)$

$x = \rho\sin\phi\cos\theta = 3\sin\frac{\pi}{2}\cos\frac{\pi}{3} = \frac{3}{2}$

$y = \rho\sin\phi\sin\theta = 3\sin\frac{\pi}{2}\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$

$z = \rho\cos\phi = 3\cos\frac{\pi}{2} = 0$

$r = \rho\sin\phi = 3\sin\frac{\pi}{2} = 3$

Rectangular: $\left(\frac{3}{2},\frac{3\sqrt{3}}{2},0\right)$

Cylindrical: $\left(3,\frac{\pi}{3},0\right)$

40. For each of the following, give the equation of the described object in rectangular, cylindrical, and spherical coordinates.

(6) a. The sphere centered at (0,0,0) with radius 1.

Rectangular: $x^2 + y^2 + z^2 = 1$ Cylindrical: $r^2 + z^2 = 1$ Spherical: $\rho = 1$

(6) b. The cylinder whose intersection with the xy-plane is the circle centered at (0,0) with radius 3.

Rectangular: $x^2 + y^2 = 9$ Cylindrical: $r = 3$ Spherical: $\rho \sin\phi = 3$

41. For each of the following, calculate the limit or show that it does not exist.

(4) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y+3}{x+y+1} = 3$

(4) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

$$\lim_{(x,0)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^4} = 0$$

$$\lim_{(\sqrt{y},y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(\sqrt{y},y)\to(0,0)} \frac{y^2}{2y^2} = \tfrac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \text{ DNE}$$

(6) 42. Use the definition of limit to show that

$$\lim_{(x,y)\to(1,-2)} \big(2x+y+1\big) = 1.$$

Proof: Let $\varepsilon > 0$ and choose δ such that $0 < \delta < \frac{\varepsilon}{3}$. Suppose that $d((x,y),(1,-2)) < \delta$. Then

$$\sqrt{(x-1)^2+(y+2)^2} < \delta$$

and so

$$\sqrt{(x-1)^2} = |x-1| < \delta$$

and

$$\sqrt{(y+2)^2} = |y+2| < \delta.$$

Hence,

$$2|x-1| + |y+2| < 3\delta$$

$$|2(x-1)+(y+2)| < 3\delta$$

$$|2x-2+y+2| < \varepsilon$$

$$|2x+y| < \varepsilon$$

$$|(2x+y+1)-1| < \varepsilon$$

as desired. ■

43. Let $f(x,y) = x^2y - y$.

(5) a. Find all local extrema and saddle points.

$f_x(x,y) = 2xy$

$f_y(x,y) = x^2 - 1$

$f_{xx}(x,y) = 2y$

$f_{yy}(x,y) = 0$

$f_{xy}(x,y) = 2x$

Critical Points: (-1,0), (1,0)

D(-1,0) = -4

D(1,0) = -4

Saddle Points: (-1,0,0), (1,0,0)

(5) b. Find absolute extrema of f on the region enclosed by the triangle with vertices (-1,0), (0,0), and (0,1).

Note that since one critical point of f is on the boundary of the triangle and the other is not in or on the triangle, we need only analyze f on the boundary of the triangle. Find the maximum and minimum values of

(i) $g(x) = f(x{,}0)$ on [-1,0];

(ii) $g(y) = f(0{,}y)$ on [0,1];

(iii) $g(x) = f(x{,}x + 1)$ on [-1,0].

(ii) Note that $g(x) = f(x{,}0) = 0$ for all $x \in$ [-1,0].

(iii) Note that $g(y) = f(0{,}y) = \text{-}y$ for all $y \in$ [0,1].

(iv) $g(x) = f(x{,}x + 1) = x^2(x + 1) - (x + 1) = x^3 + x^2 - x - 1$

$g'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$

Critical Numbers: $\frac{1}{3} \notin$ [-1,0], -1 $\in$ [-1,0]

$g(\text{-}1) = 0$

Absolute Max: 0 at $(x{,}0)$ for all $x \in$ [-1,0]

Absolute Min: -1 at (0,1)

(8) 44. Prove that $f(x,y) = 2x + y^2 + 1$ is differentiable for all $(a,b) \in \mathbb{R}^2$ and find $f'_{(a,b)}(x,y)$.

***Proof*:** Note that

$f_x(x,y) = 2$

and

$f_y(x,y) = 2y.$

Consider

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - \big(f_x(a,b)h + f_y(a,b)k\big)}{\|(h,k)\|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{2(a+h) + (b+k)^2 + 1 - (2a + b^2 + 1) - (2h + 2bk)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{2a + 2h + b^2 + 2bk + k^2 + 1 - 2a - b^2 - 1 - 2h - 2bk}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2+k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{k^2}{|k|}$$

$$= \lim_{(h,k)\to(0,0)} |k|$$

$$= 0$$

as desired. ■

$f'_{(a,b)}(x,y) = 2x + 2by$

Section 1.4: Exam 34

Exam 1 Math 2673 Spring 2004

Name:__________________________

E-mail (optional):__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

45. For each of the following, describe the region in space being defined.

(4) a. $-6 \leq x^2 - 2x + y^2 + 6y + z^2 \leq -1$

(4) b. $y^2 + z^2 = 9$

46. Let $\boldsymbol{v} = \langle 1,-1,4 \rangle$ and $\boldsymbol{w} = \langle 2,2,0 \rangle$. Find the following.

(2) a. $2\boldsymbol{v} - \boldsymbol{w} =$

(2) b. $\boldsymbol{v} \cdot \boldsymbol{w} =$

(3) c. $\|\boldsymbol{v}\| =$

(3) d. Find the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

47. Let $\boldsymbol{v} = \langle 1,1,1 \rangle$ and $\boldsymbol{w} = \langle 1,1,0 \rangle$.

(3) a. $\text{comp}_{\boldsymbol{v}} \boldsymbol{w} =$

(3) b. $\text{proj}_{\boldsymbol{v}} \boldsymbol{w} =$

(3) c. $\boldsymbol{v} \times \boldsymbol{w} =$

(4) 48. Find the area of the parallelogram determined by $\langle 1,2,3 \rangle$ and $\langle -2,4,1 \rangle$.

(4) 49. Find the volume of the parallelpiped determined by $\langle -3,0,5 \rangle$, $\langle 1,2,3 \rangle$, and $\langle -2,4,1 \rangle$.

(4) 50. Find the equation of the plane that contains the points (1,-1,-1), (2,0,1), and (4,-2,3).

(3) 51. Find equations of the line containing the point (1,-1,0) in the direction of $\langle$-3,1,1$\rangle$.

(4) 52. Does the line with parametric equations $x = 4t + 1$, $y = 2t - 3$, and $z = t + 2$ intersect the plane with equation $2(x - 5) + y + 1 + 6(z - 3) = 0$? If so, where?

(5) 53. Find an equation of the line where the following planes intersect.

$2x - y + 3z = 1$

-$3x + y + 5z = 7$

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54. For each of the following, describe the region in space being defined.

(4) a. $-6 \leq x^2 - 2x + y^2 + 6y + z^2 \leq -1$

$-6 \leq (x-1)^2 - 1 + (y+3)^2 - 9 + z^2 \leq -1$

$4 \leq (x-1)^2 + (y+3)^2 + z^2 \leq 9$

This inequality describes the set of all points on or in the sphere centered at (1,-3,0) with radius 3 that are not in the sphere centered at $(1, -3, 0)$ with radius 2.

(4) b. $y^2 + z^2 = 9$

This is the circular cylinder with radius 3 and having the x-axis as its central axis.

55. Let $\boldsymbol{v} = \langle 1,-1,4 \rangle$ and $\boldsymbol{w} = \langle 2,2,0 \rangle$. Find the following.

(2) a. $2\boldsymbol{v} - \boldsymbol{w} = \langle 2-2, -2-2, 8-0 \rangle = \langle 0,-4,8 \rangle$

(2) b. $\boldsymbol{v} \cdot \boldsymbol{w} = 2 - 2 + 0 = 0$

(3) c. $\|\boldsymbol{v}\| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

(3) d. Find the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Let θ be the angle between and $\boldsymbol{v}$ and $\boldsymbol{w}$. Since $\boldsymbol{v} \cdot \boldsymbol{w} = 0$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal.

56. Let $\boldsymbol{v} = \langle 1,1,1 \rangle$ and $\boldsymbol{w} = \langle 1,1,0 \rangle$.

(3) a. $\text{comp}_{\boldsymbol{v}} \boldsymbol{w} = \dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|} = \dfrac{2}{\sqrt{3}}$

(3) b. $\text{proj}_{\boldsymbol{v}} \boldsymbol{w} = \left(\dfrac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|} \right) \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \langle \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

(3) c. $\boldsymbol{v} \times \boldsymbol{w} = \langle -1,1,0 \rangle$

(4) 57. Find the area of the parallelogram determined by $\langle 1,2,3 \rangle$ and $\langle -2,4,1 \rangle$.

$\|\langle 1,2,3 \rangle \times \langle -2,4,1 \rangle\| = \|\langle -10,-7,8 \rangle\| = \sqrt{213}$

(4) 58. Find the volume of the parallelpiped determined by $\langle -3,0,5 \rangle$, $\langle 1,2,3 \rangle$, and $\langle -2,4,1 \rangle$.

$|\langle -3,0,5 \rangle \cdot (\langle 1,2,3 \rangle \times \langle -2,4,1 \rangle)| = |\langle -3,0,5 \rangle \cdot \langle -10,-7,8 \rangle| = 70$

(4) 59. Find the equation of the plane that contains the points (1,-1,-1), (2,0,1), and (4,-2,3).

Let $\boldsymbol{v} = \langle 1, 1, 2\rangle$, $\boldsymbol{w} = \langle 3, \text{-}1, 4\rangle$, and $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle 6,2,\text{-}4\rangle$. The equation of the plane with normal vector $\langle 6,2,\text{-}4\rangle$ and containing the point (1,-1,-1) is $6(x-1)+2(y+1)-4(z+1) = 0$.

(3) 60. Find equations of the line containing the point (1,-1,0) in the direction of $\langle \text{-}3,1,1\rangle$.

$x = \text{-}3t + 1$ $\quad$ $y = t - 1$ $\quad$ $z = t$

(4) 61. Does the line with parametric equations $x = 4t + 1$, $y = 2t - 3$, and $z = t + 2$ intersect the plane with equation $2(x - 5) + y + 1 + 6(z - 3) = 0$? If so, where?

$2[(4t + 1) - 5] + 2t - 3 + 1 + 6[(t + 2) - 3] = 0$

$16t - 16 = 0$

$t = 1$

The plane and the line intersect at the point $(5, \text{-}1, 3)$.

(5) 62. Find an equation of the line where the following planes intersect.

$2x - y + 3z = 1$

$\text{-}3x + y + 5z = 7$

Let $z = 0$.

$$\begin{array}{rcrcr} 2x & - & y & = & 1 \\ \text{-}3x & + & y & = & 7 \end{array}$$

$\text{-}x = 8$

$x = \text{-}8$

$y = \text{-}17$

So the point (-8,-17,0) is on the line of intersection of the two planes.

Normal vectors of the two planes are $\boldsymbol{n}_1 = \langle 2,\text{-}1,3\rangle$ and $\boldsymbol{n}_2 = \langle \text{-}3,1,5\rangle$. So a vector parallel to the line of intersection is $\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle \text{-}8,\text{-}19,\text{-}1\rangle$. Equations of the line passing through the point (-8,-17,0) with parallel vector $\langle \text{-}8,\text{-}19,\text{-}1\rangle$ are

$x = \text{-}8 - 8t$

$y = \text{-}17 - 19t$

$z = \text{-}t$.

Section 1.5: Final

Final Exam Math 2673 Summer 2004

Name:____________________

E-mail (optional):____________________

Directions: Show all of your work in the space provided and justify all of your answers. You may use written materials but you are not permitted to ask another person for help (except for the instructor).

63. (2) a. Find a vector-valued function whose graph is the line connecting the points

(1,-1,2) and (5,1,6). (1,0,2) and (1,2,6). (2,-3,1) and (0,3,5). (5,4,-2) and (1,-2,3). (2,3,2) and (-1,2,5). (2,0,8) and (0,2,7). (3,-5,3) and (2,1,6). (2,-3,1) and (-1,-2,-5). (-1,1,-2) and (3,3,9). (2,2,6) and (5,1,4). (0,3,9) and (0,-6,2). (8,8,7) and (7,5,4). (3,3,3) and (-1,0,6).

(2) b. Parameterize the curve by arc length.

(2) 64. For each of the following, determine whether or not the lines are coplanar. If so, give the equation of the plane. If not, explain why they are not.

$$x = t + 1,\quad y = t - 2,\quad z = 3t + 1$$
$$x = 2s + 2,\quad y = 3s + 1,\quad z = s + 2$$

$$x = 2t - 3,\quad y = -t - 2,\quad z = 2t + 3$$
$$x = 2s + 2,\quad y = s - 3,\quad z = s + 2$$

$$x = -2t + 1,\quad y = 3t,\quad z = t - 4$$
$$x = 3s + 1,\quad y = -s - 3,\quad z = 2s - 1$$

$$x = -t - 2,\quad y = 2t + 6,\quad z = 3t + 1$$
$$x = 2s - 2,\quad y = -s,\quad z = -s + 3$$

$$x = t + 1,\quad y = t - 2,\quad z = 3t + 1$$
$$x = 2s + 2,\quad y = 3s + 1,\quad z = s + 2$$

$$x = 2t - 3,\quad y = -t - 2,\quad z = 2t + 3$$
$$x = 2s + 2,\quad y = s - 3,\quad z = s + 2$$

$$x = -2t + 1,\quad y = 3t,\quad z = t - 4$$
$$x = 3s + 1,\quad y = -s - 3,\quad z = 2s - 1$$

$$x = -t - 2,\quad y = 2t + 6,\quad z = 3t + 1$$
$$x = 2s - 2,\quad y = -s,\quad z = -s + 3$$

$$\begin{aligned} x &= t + 1 \\ y &= t - 2 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= 3s + 1 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= 2t - 3 \\ y &= \text{-}t - 2 \\ z &= 2t + 3 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= s - 3 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= \text{-}2t + 1 \\ y &= 3t \\ z &= t - 4 \end{aligned}$$

$$\begin{aligned} x &= 3s + 1 \\ y &= \text{-}s - 3 \\ z &= 2s - 1 \end{aligned}$$

$$\begin{aligned} x &= \text{-}t - 2 \\ y &= 2t + 6 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s - 2 \\ y &= \text{-}s \\ z &= \text{-}s + 3 \end{aligned}$$

$$\begin{aligned} x &= t + 1 \\ y &= t - 2 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= 3s + 1 \\ z &= s + 2 \end{aligned}$$

(2) 65. Do the following planes intersect? If so, find the line of intersection.

$$\begin{aligned} 2x + 3y - z &= 7 \\ 3x - 8y + 2z &= 11 \end{aligned}$$

$$\begin{aligned} 2x + 2y - z &= 7 \\ 2x - 8y + 2z &= 10 \end{aligned}$$

$$\begin{aligned} x + y - z &= 6 \\ 2x + 3y + z &= 8 \end{aligned}$$

$$\begin{aligned} 2x - y + z &= -7 \\ 3x + 3y - z &= 6 \end{aligned}$$

$$\begin{aligned} x + y + 3z &= -1 \\ x + 2y - 2z &= 8 \end{aligned}$$

$$\begin{aligned} 3x - y - 2z &= -3 \\ x + y + z &= 7 \end{aligned}$$

$$\begin{aligned} 2x + 3y - z &= 7 \\ 3x - 8y + 2z &= 11 \end{aligned}$$

$$\begin{aligned} 2x + 3y - z &= 7 \\ 3x - 8y + 2z &= 11 \end{aligned}$$

$$\begin{aligned} 2x + 2y - z &= 7 \\ 2x - 8y + 2z &= 10 \end{aligned}$$

$$\begin{aligned} x + y - z &= 6 \\ 2x + 3y + z &= 8 \end{aligned}$$

$$\begin{aligned} 2x - y + z &= -7 \\ 3x + 3y - z &= 6 \end{aligned}$$

$$\begin{aligned} x + y + 3z &= -1 \\ x + 2y - 2z &= 8 \end{aligned}$$

$$\begin{aligned} 3x - y - 2z &= -3 \\ x + y + z &= 7 \end{aligned}$$

66. For each of the following, sketch the graph of the function and use arrows to indicate the direction of increasing t.

(2) a. $\boldsymbol{r}(t) = \langle t, \sin t\rangle$

$\boldsymbol{r}(t) = \langle e^t, t\rangle$

$\boldsymbol{r}(t) = \langle \sin t, t\rangle$

$\boldsymbol{r}(t) = \langle t, e^t\rangle$

$\boldsymbol{r}(t) = \langle \cos t, t\rangle$

$\boldsymbol{r}(t) = \langle t, \cos t\rangle$

$\boldsymbol{r}(t) = \langle t, \ln t\rangle$

$\boldsymbol{r}(t) = \langle \ln t, t\rangle$

$\boldsymbol{r}(t) = \langle t, \tan t\rangle$

$\boldsymbol{r}(t) = \langle \tan t, t\rangle$

$\boldsymbol{r}(t) = \langle \sin t, t\rangle$

$\boldsymbol{r}(t) = \langle \ln t, t\rangle$

$\boldsymbol{r}(t) = \langle t, \sin t\rangle$

(2) b.

$\boldsymbol{r}(t) = \langle 3\cos t, 2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -3\cos t, 2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 3\cos t, -2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -3\cos t, -2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle = -2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, -3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -2\cos t, -3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 3\cos t, 2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, -3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -2\cos t, -3\sin t, t\rangle$

(4) 67. Let $\boldsymbol{f}(t) = \left\langle e^t \cos t, \sqrt{t^2+1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle e^t \sin t, \sqrt{t^2+1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle e^t \cos t, \sqrt{t^2-1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle e^t \sin t, \sqrt{t^2-1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle e^t \cos t, \sqrt{1-t^2}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle e^t \sin t, \sqrt{1-t^2}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle \ln t, \sqrt{t^2+1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle \ln t, \sqrt{t^2-1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle \ln t, \sqrt{1-t^2}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle \sec^3 t, \sqrt{t^2+1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle \sec^3 t, \sqrt{t^2-1}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle \sec^3 t, \sqrt{1-t^2}\right\rangle$.

$\boldsymbol{f}(t) = \left\langle e^t \cos t, \sqrt{t^2+1}\right\rangle$.

$$\int_1^2 \boldsymbol{f}(t)\,dt =$$

(2) 68. Sketch the graph of $f(x,y) = e^x$.

$f(x,y) = e^y$.

$f(x,y) = \sin x$.

$f(x,y) = \sin y$.

$f(x,y) = \cos x$.

$f(x,y) = \cos y$.

$f(x,y) = \ln x$.

$f(x,y) = \ln y$.

$f(x,y) = x^2$.

$f(x,y) = y^2$.

$f(x,y) = x + y$.

$f(x,y) = x - y$.

$f(x,y) = y - x$.

(2) 69. Find the following limit or show that it does not exist.

$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$
$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$
$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$ $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$

(2) 70. Use the ε–δ definition of limit to prove $\lim_{(x,y)\to(1,1)} \big(3x+2y\big) = 5$. $\lim_{(x,y)\to(2,1)} \big(3x+2y\big) = 8$. $\lim_{(x,y)\to(-1,1)} \big(3x+2y\big) = -1$. $\lim_{(x,y)\to(1,-1)} \big(3x+2y\big) = 1$. $\lim_{(x,y)\to(2,2)} \big(3x+2y\big) = 10$. $\lim_{(x,y)\to(0,1)} \big(3x+2y\big) = 2$. $\lim_{(x,y)\to(1,0)} \big(3x+2y\big) = 3$. $\lim_{(x,y)\to(2,1)} \big(2x+3y\big) = 7$. $\lim_{(x,y)\to(2,1)} \big(x+3y\big) = 5$. $\lim_{(x,y)\to(1,1)} \big(5x-2y\big) = 3$. $\lim_{(x,y)\to(-2,1)} \big(3x+2y\big) = -4$. $\lim_{(x,y)\to(0,1)} \big(2x+3y\big) = 3$. $\lim_{(x,y)\to(2,-1)} \big(3x+2y\big) = 4$.

(3) 71. Use the $\varepsilon - \delta$ definition of limit to prove $\lim_{(x,y)\to(1,1)} xy = 1$. $\lim_{(x,y)\to(2,1)} xy = 2$. $\lim_{(x,y)\to(1,2)} xy = 2$. $\lim_{(x,y)\to(2,2)} xy = 4$. $\lim_{(x,y)\to(3,1)} xy = 3$. $\lim_{(x,y)\to(1,3)} xy = 3$. $\lim_{(x,y)\to(2,3)} xy = 6$. $\lim_{(x,y)\to(3,2)} xy = 6$. $\lim_{(x,y)\to(4,1)} xy = 4$. $\lim_{(x,y)\to(1,4)} xy = 4$. $\lim_{(x,y)\to(2,4)} xy = 8$. $\lim_{(x,y)\to(4,2)} xy = 8$. $\lim_{(x,y)\to(4,3)} xy = 12$.

(3) 72. Let $f(x,y,z) = x^2y + 4y + 3z^2$. Show that f is differentiable at (a,b,c) for all $(a,b,c) \in \mathbb{R}^3$ and find $f'_{(a,b,c)}(x,y,z)$.

73. Let $f(x,y) = 3x^2y + 3xy + y^2$. $f(x,y) = x^2y + xy + y^2$. $f(x,y) = 2x^2y + 2xy + y^2$. $f(x,y) = 4x^2y + 4xy + y^2$. $f(x,y) = 5x^2y + 5xy + y^2$. $f(x,y) = 3x^2y + 3xy + y^2$. $f(x,y) = x^2y + xy + y^2$. $f(x,y) = 2x^2y + 2xy + y^2$. $f(x,y) = 4x^2y + 4xy + y^2$. $f(x,y) = 5x^2y + 5xy + y^2$. $f(x,y) = 3x^2y + 3xy + y^2$. $f(x,y) = x^2y + xy + y^2$. $f(x,y) = 2x^2y + 2xy + y^2$.

(3) a. Find all local extrema and saddle points of f.

(2) b. Find all extrema of f on the triangle with vertices (0,0), (1,0), and (1,1).

(2) 74. Let R = [1,2] × [0,1] R = [1,2] × [1,1] R = [-1,1] × [1,2] R = [0,1] × [1,2] R = [0,1] × [1,1] R = [-2,0] × [0,1] R = [-2,1] × [0,1] R = [-2,1] × [1,3] R = [0,1] × [2,3] R = [-1,0] × [1,2] R = [1,2] × [0,1] R = [1,2] × [1,1] R = [-1,1] × [1,2] and calculate $\iint_R (x^2y + y^3)\,dA$.

(2) 75. Let C be the circle centered at (0,0) with radius 1 $\sqrt{2}$ 2 $\sqrt{3}$ 3 4 5 $\sqrt{5}$ 6 7 $\sqrt{7}$ 8 9 and calculate $\iint_C (xe^{x^2+y^2})\,dA$.

(2) 76. Let T be the triangle with vertices (0,0), (4,0), and (4,1) and calculate $\iint_T (x^2y + y^3)\,dA$.

77. Let $f(x,y,z) = x^2y + 3xy + yz^2$. Let R be the rectangular prism determined by $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. Let T be the tetrahedron with vertices (0,0,0), (2,0,0), (0,1,0), and (0,0,3). Let C be the right circular cylinder whose base is the unit circle and whose height is 5. Finally, let S be the unit sphere.

(2) a. $\iiint_R f(x,y,z)\,dV$

(3) b. $\iiint_T f(x,y,z)\,dV$

(3) c. $\iiint_C f(x,y,z)\,dV$

(3) d. $\iiint_S f(x,y,z)\,dV$

Section 1.6: Final1

Final Exam Math 2673 Summer 2004

Name:__________________________

E-mail (optional):__________________________

Directions: Show all of your work in the space provided and justify all of your answers. You may use written materials but you are not permitted to ask another person for help (except for the instructor).

78. (2) a. Find a vector-valued function whose graph is the line connecting the points (1,-1,2) and (5,1,6). (1,0,2) and (1,2,6). (2,-3,1) and (0,3,5). (5,4,-2) and (1,-2,3). (2,3,2) and (-1,2,5). (2,0,8) and (0,2,7). (3,-5,3) and (2,1,6). (2,-3,1) and (-1,-2,-5). (-1,1,-2) and (3,3,9). (2,2,6) and (5,1,4). (0,3,9) and (0,-6,2). (8,8,7) and (7,5,4). (3,3,3) and (-1,0,6).

(2) b. Parameterize the curve by arc length.

(3) 79. For each of the following, determine whether or not the lines are coplanar. If so, give the equation of the plane. If not, explain why they are not.

$$\begin{aligned} x &= t + 1 \\ y &= t - 2 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= 3s + 1 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= 2t - 3 \\ y &= -t - 2 \\ z &= 2t + 3 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= s - 3 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= -2t + 1 \\ y &= 3t \\ z &= t - 4 \end{aligned}$$

$$\begin{aligned} x &= 3s + 1 \\ y &= -s - 3 \\ z &= 2s - 1 \end{aligned}$$

$$\begin{aligned} x &= -t - 2 \\ y &= 2t + 6 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s - 2 \\ y &= -s \\ z &= -s + 3 \end{aligned}$$

$$\begin{aligned} x &= t + 1 \\ y &= t - 2 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= 3s + 1 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= 2t - 3 \\ y &= -t - 2 \\ z &= 2t + 3 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= s - 3 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= -2t + 1 \\ y &= 3t \\ z &= t - 4 \end{aligned}$$

$$\begin{aligned} x &= 3s + 1 \\ y &= -s - 3 \\ z &= 2s - 1 \end{aligned}$$

$$\begin{aligned} x &= -t - 2 \\ y &= 2t + 6 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s - 2 \\ y &= -s \\ z &= -s + 3 \end{aligned}$$

$$\begin{aligned} x &= t + 1 \\ y &= t - 2 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= 3s + 1 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= 2t - 3 \\ y &= -t - 2 \\ z &= 2t + 3 \end{aligned}$$

$$\begin{aligned} x &= 2s + 2 \\ y &= s - 3 \\ z &= s + 2 \end{aligned}$$

$$\begin{aligned} x &= -2t + 1 \\ y &= 3t \\ z &= t - 4 \end{aligned}$$

$$\begin{aligned} x &= 3s + 1 \\ y &= -s - 3 \\ z &= 2s - 1 \end{aligned}$$

$$\begin{aligned} x &= -t - 2 \\ y &= 2t + 6 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{aligned} x &= 2s - 2 \\ y &= -s \\ z &= -s + 3 \end{aligned}$$

$$\begin{aligned} x &= t + 1 \\ y &= t - 2 \\ z &= 3t + 1 \end{aligned}$$

$$\begin{array}{rcrcr} x &=& 2s &+& 2 \\ y &=& 3s &+& 1 \\ z &=& s &+& 2 \end{array}$$

(3) 80. Do the following planes intersect? If so, find the line of intersection.

$$\begin{array}{rcrcrcr} 2x &+& 3y &-& z &=& 7 \\ 3x &-& 8y &+& 2z &=& 11 \end{array}$$

$$\begin{array}{rcrcrcr} 2x &+& 2y &-& z &=& 7 \\ 2x &-& 8y &+& 2z &=& 10 \end{array}$$

$$\begin{array}{rcrcrcr} x &+& y &-& z &=& 6 \\ 2x &+& 3y &+& z &=& 8 \end{array}$$

$$\begin{array}{rcrcrcr} 2x &-& y &+& z &=& -7 \\ 3x &+& 3y &-& z &=& 6 \end{array}$$

$$\begin{array}{rcrcrcr} x &+& y &+& 3z &=& -1 \\ x &+& 2y &-& 2z &=& 8 \end{array}$$

$$\begin{array}{rcrcrcr} 3x &-& y &-& 2z &=& -3 \\ x &+& y &+& z &=& 7 \end{array}$$

$$\begin{array}{rcrcrcr} 2x &+& 3y &-& z &=& 7 \\ 3x &-& 8y &+& 2z &=& 11 \end{array}$$

$$\begin{array}{rcrcrcr} 2x &+& 3y &-& z &=& 7 \\ 3x &-& 8y &+& 2z &=& 11 \end{array}$$

$$\begin{array}{rcrcrcr} 2x &+& 2y &-& z &=& 7 \\ 2x &-& 8y &+& 2z &=& 10 \end{array}$$

$$\begin{array}{rcrcrcr} x &+& y &-& z &=& 6 \\ 2x &+& 3y &+& z &=& 8 \end{array}$$

$$\begin{array}{rcrcrcr} 2x &-& y &+& z &=& -7 \\ 3x &+& 3y &-& z &=& 6 \end{array}$$

$$\begin{array}{rcrcrcr} x &+& y &+& 3z &=& -1 \\ x &+& 2y &-& 2z &=& 8 \end{array}$$

$$\begin{array}{rcrcrcr} 3x &-& y &-& 2z &=& -3 \\ x &+& y &+& z &=& 7 \end{array}$$

81. For each of the following, sketch the graph of the function and use arrows to indicate the direction of increasing t.

(3) a. $\boldsymbol{r}(t) = \langle e^t, t\rangle$ $\qquad\qquad$ $\boldsymbol{r}(t) = \langle t, e^t\rangle$

$\boldsymbol{r}(t) = \langle t, \sin t\rangle$

$\boldsymbol{r}(t) = \langle \sin t, t\rangle$

$\boldsymbol{r}(t) = \langle \cos t, t\rangle$

$\boldsymbol{r}(t) = \langle t, \cos t\rangle$

$\boldsymbol{r}(t) = \langle t, \ln t\rangle$

$\boldsymbol{r}(t) = \langle \ln t, t\rangle$

$\boldsymbol{r}(t) = \langle t, \tan t\rangle$

$\boldsymbol{r}(t) = \langle \tan t, t\rangle$

$\boldsymbol{r}(t) = \langle \sin t, t\rangle$

$\boldsymbol{r}(t) = \langle \ln t, t\rangle$

$\boldsymbol{r}(t) = \langle t, \sin t\rangle$

(3) b.

$\boldsymbol{r}(t) = \langle 3\cos t, 2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -3\cos t, 2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 3\cos t, -2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -3\cos t, -2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle = -2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, -3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -2\cos t, -3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 3\cos t, 2\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -2\cos t, 3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle 2\cos t, -3\sin t, t\rangle$

$\boldsymbol{r}(t) = \langle -2\cos t, -3\sin t, t\rangle$

(3) 82. Let $\boldsymbol{f}(t) = \left\langle e^t\cos t, \sqrt{t^2+1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle e^t\sin t, \sqrt{t^2+1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle e^t\cos t, \sqrt{t^2-1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle e^t\sin t, \sqrt{t^2-1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle e^t\cos t, \sqrt{1-t^2}\right\rangle$. $\boldsymbol{f}(t) = \left\langle e^t\sin t, \sqrt{1-t^2}\right\rangle$. $\boldsymbol{f}(t) = \left\langle \ln t, \sqrt{t^2+1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle \ln t, \sqrt{t^2-1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle \ln t, \sqrt{1-t^2}\right\rangle$. $\boldsymbol{f}(t) = \left\langle \sec^3 t, \sqrt{t^2+1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle \sec^3 t, \sqrt{t^2-1}\right\rangle$. $\boldsymbol{f}(t) = \left\langle \sec^3 t, \sqrt{1-t^2}\right\rangle$. $\boldsymbol{f}(t) = \left\langle e^t\cos t, \sqrt{t^2+1}\right\rangle$.

$$\int_1^2 \boldsymbol{f}(t)\,dt =$$

83. A projectile is launched with initial velocity vector $\boldsymbol{v} = \langle 30,40,50 \rangle$. The wind is blowing southeast at a rate of 50 ft/sec.

(4) a. Find the vector functions that describe velocity and motion.

(2) b. Find the maximum height.

(2) c. Find the position of impact.

(2) d. Find the speed of impact.

(2) 84. Sketch the graph of $f(x,y) = e^x$. $f(x,y) = e^y$. $f(x,y) = \sin x$. $f(x,y) = \sin y$. $f(x,y) = \cos x$. $f(x,y) = \cos y$. $f(x,y) = \ln x$. $f(x,y) = \ln y$. $f(x,y) = x^2$. $f(x,y) = y^2$. $f(x,y) = x + y$. $f(x,y) = x - y$. $f(x,y) = y - x$.

(6) 85. For each of the six quadric surfaces we studied, give an example using irrational coefficients and sketch the graph.

(3) 86. Find the following limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$$
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}$$
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x} \quad \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{y}$$

(3) 87. Use the ε–δ defintion of limit to prove $\lim_{(x,y)\to(1,1)} \big(3x+2y\big) = 5$. $\lim_{(x,y)\to(2,1)} \big(3x+2y\big) = 8$. $\lim_{(x,y)\to(-1,1)} \big(3x+2y\big) = -1$. $\lim_{(x,y)\to(1,-1)} \big(3x+2y\big) = 1$. $\lim_{(x,y)\to(2,2)} \big(3x+2y\big) = 12$. $\lim_{(x,y)\to(0,1)} \big(3x+2y\big) = 1$. $\lim_{(x,y)\to(1,0)} \big(3x+2y\big) = 3$. $\lim_{(x,y)\to(2,1)} \big(2x+3y\big) = 5$. $\lim_{(x,y)\to(2,1)} \big(x+3y\big) = 5$. $\lim_{(x,y)\to(1,1)} \big(5x-2y\big) = 3$. $\lim_{(x,y)\to(-2,1)} \big(3x+2y\big) = -4$. $\lim_{(x,y)\to(0,1)} \big(2x+3y\big) = 3$. $\lim_{(x,y)\to(2,-1)} \big(3x+2y\big) = 4$.

(3) 88. Use the $\varepsilon - \delta$ defintion of limit to prove $\lim_{(x,y)\to(1,1)} xy = 1$. $\lim_{(x,y)\to(2,1)} xy = 2$. $\lim_{(x,y)\to(1,2)} xy = 2$. $\lim_{(x,y)\to(2,2)} xy = 4$. $\lim_{(x,y)\to(3,1)} xy = 3$. $\lim_{(x,y)\to(1,3)} xy = 3$. $\lim_{(x,y)\to(2,3)} xy = 6$. $\lim_{(x,y)\to(3,2)} xy = 6$. $\lim_{(x,y)\to(4,1)} xy = 4$. $\lim_{(x,y)\to(1,4)} xy = 4$. $\lim_{(x,y)\to(2,4)} xy = 8$. $\lim_{(x,y)\to(4,2)} xy = 8$. $\lim_{(x,y)\to(4,3)} xy = 12$.

(3) 89. Let $f(x,y,z) = x^2y + 4y + 3z^2$. Show that f is differentiable at (a,b,c) for all $(a,b,c) \in \mathbb{R}^3$ and find $f'_{(a,b,c)}(x,y,z)$.

Let $f(x,y,z) = 3x^2y + 3xy + y^2$. $f(x,y,z) = x^2y + xy + y^2$. $f(x,y,z) = 2x^2y + 2xy + y^2$. $f(x,y,z) = 4x^2y + 4xy + y^2$. $f(x,y,z) = 5x^2y + 5xy + y^2$. $f(x,y,z) = 3x^2y + 3xy + y^2$. $f(x,y,z) = x^2y+xy+y^2$. $f(x,y,z) = 2x^2y+2xy+y^2$. $f(x,y,z) = 4x^2y+4xy+y^2$. $f(x,y,z) = 5x^2y + 5xy + y^2$. $f(x,y,z) = 3x^2y + 3xy + y^2$. $f(x,y,z) = x^2y + xy + y^2$. $f(x,y,z) = 2x^2y + 2xy + y^2$.

(3) a. Find all local extrema and saddle points of f.

(3) b. Find all extrema of f on the triangle with vertices (0,0), (1,0), and (1,1).

(2) 90. Let R = [1,2] × [0,1] R = [1,2] × [1,1] R = [-1,1] × [1,2] R = [0,1] × [1,2] R = [0,1] × [1,1] R = [-2,0] × [0,1] R = [-2,1] × [0,1] R = [-2,1] × [1,3] R = [0,1] × [2,3] R = [-1,0] × [1,2] R = [1,2] × [0,1] R = [1,2] × [1,1] R = [-1,1] × [1,2] and calculate $\iint_R (x^2y + y^3)\,dA$.

(3) 91. Let C be the circle centered at (0,0) with radius 1 $\sqrt{2}$ 2 $\sqrt{3}$ 3 4 5 $\sqrt{5}$ 6 7 $\sqrt{7}$ 8 9 and calculate $\iint_C (xe^{x^2+y^2})\,dA$.

(3) 92. Let T be the triangle with vertices (0,0), 4,0), and 4,1) and calculate $\iint_T (x^2y + y^3)\,dA$.

93. Let $f(x,y,z) = x^2y + 3xy + yz^2$. Let R be the rectangular prism determined by $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$. Let T be the tetrahedron with vertices (0,0,0), (2,0,0), (0,1,0), and (0,0,3). Let C be the right circular cylinder whose base is the unit circle and whose height is 5. Finally, let S be the unit sphere.

(2) a. $\iiint_R f(x,y,z)\,dV$

(3) b. $\iiint_R f(x,y,z)\,dT$

(3) c. $\iiint_C f(x,y,z)\,dT$

(3) d. $\iiint_S f(x,y,z)\,dT$

Chapter 2: Spring 2015

Section 2.1: Quizzes

Quiz 1 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Describe the region in space being defined.

-6 $\leq x^2 - 2x + y^2 + 6y + z^2 \leq$ -1

-6 $\leq (x-1)^2 - 1 + (y+3)^2 - 9 + z^2 \leq$ -1

$4 \leq (x-1)^2 + (y+3)^2 + z^2 \leq 9$

This inequality describes the set of all points on or in the sphere centered at (1,-3,0) with radius 3 that are not in the sphere centered at $(1, \text{-}3, 0)$ with radius 2.

(Page 815: 43) **(1) 2.** Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

Quiz 2 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{v} = \langle 1,2,\text{-}1\rangle$, $\boldsymbol{w} = \langle 0,1,1\rangle$, and θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(1) a. Find $\cos\theta$.

$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{1}{\sqrt{6}\cdot\sqrt{2}} = \frac{1}{2\sqrt{3}}$

(1) b. Find $\boldsymbol{v}\times\boldsymbol{w}$.

$\boldsymbol{v}\times\boldsymbol{w} = \langle 3,\text{-}1,1\rangle$

Quiz 3 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the equation of the plane that contains the points (1,1,0), (2,-1,1), and (2,-1,3).

Let $\boldsymbol{v} = \langle 0,0,2 \rangle$, $\boldsymbol{w} = \langle 1,\text{-}2,1 \rangle$, and $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle 4,2,0 \rangle$.

The equation of the plane containing the point (1,1,0) with normal vector $\boldsymbol{n} = \langle 2,1,0 \rangle$ is $2(x-1) + 1(y-1) = 0$.

Quiz 4 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let l_1 and l_2 be the lines with parametric equations given below.

$x = 3t - 1$ $\qquad$ $x = 4s + 1$

$y = t + 6$ $\qquad$ $y = 3s + 5$

$z = 2t + 2$ $\qquad$ $z = \text{-}s + 7$

(1) a. Do l_1 and l_2 intersect? If so, where?

$$\begin{array}{ccccccc} 3t & - & 1 & = & 4s & + & 1 \\ t & + & 6 & = & 3s & + & 5 \end{array}$$

$$\begin{array}{ccccc} 3t & - & 4s & = & 2 \\ t & - & 3s & = & \text{-}1 \end{array}$$

$5s = 5$

$s = 1$

$t = 2$

The lines intersect at (5,8,6).

(2) b. Find the equation of the plane containing l_1 and l_2.

A vector in the direction of l_1 is $\boldsymbol{v} = \langle 3,1,2 \rangle$ and a vector in the direction of l_2 is $\boldsymbol{w} = \langle 4,3,\text{-}1 \rangle$. So a normal vector to the plane is $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle \text{-}7,11,5 \rangle$. The equation of the plane with normal vector $\langle \text{-}7,11,5 \rangle$ and containing the point (5,8,6) is $\text{-}7(x+1)+11(y-6)+5(z-2) = 0$.

Quiz 5 Name:________________________

Directions: Show all of your work and justify all of your answers.

(Page 895: 21) **(2) 1.** A ball is thrown eastward into the air from the origin (in the direction of the positive x-axis). The initial velocity is $50\boldsymbol{i} + 80\boldsymbol{k}$, with speed measured in feet per second. The spin of the ball results in a southward acceleration of 4 ft/sec^2, so the acceleration vector is $\boldsymbol{a} = \text{-}4\boldsymbol{j} - 32\boldsymbol{k}$. Where does the ball land?

Quiz 6 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the domain and range. Sketch or describe the graph.

a. $f(x,y) = 1 - x^2 - y^2$

D: $\mathbb{R}^2$

R: $(-\infty,1]$

$z = 1 - x^2 - y^2$

$z - 1 = -x^2 - y^2$

$-(z - 1) = x^2 + y^2$

Paraboloid opening downward.

(1) 2. Sketch or describe the graph.

$f(x,y) = \sqrt{1 - x^2 - y^2}$

D: $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

R: $[0,1]$

$z = \sqrt{1 - x^2 - y^2}$

$z^2 = 1 - x^2 - y^2$

$x^2 + y^2 + z^2 = 1$

Top half of the unit sphere.

Quiz 7 Name:________________________

Directions: Show all of your work and justify all of your answers.

For each of the following, find the limit or prove that it does not exist.

(1) 1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2}$

$$\lim_{(x,0)\to(0,0)} \frac{4x\cdot 0}{3x^2+0^2} = 0$$

$$\lim_{(x,x)\to(0,0)} \frac{4x^2}{3x^2+x^2} = \lim_{(x,x)\to(0,0)} \frac{4x^2}{4x^2} = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} \quad \text{DNE}$$

(1) 2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y+3}{x+y+1} = 3$

(1) 3. Find the limit and use the definition of limit to prove your answer.

$$\lim_{(x,y)\to(1,1)} \big(2x+y\big) = 3$$

Proof: Let $\varepsilon > 0$ and choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{3}$. Suppose that $d((x,y),(1,1)) < \delta$. Then

$$\sqrt{(x-1)^2+(y-1)^2} < \delta$$

which implies

$$\sqrt{(x-1)^2} = |x-1| < \delta$$

and

$$\sqrt{(y-1)^2} = |y-1| < \delta.$$

So

$$2|x-1| + |y-1| < 3\delta$$

$$2|x-1| + |y-1| < \varepsilon$$

$$|2(x-1) + y - 1| < \varepsilon$$

$$|2x - 2 + y - 1| < \varepsilon$$

$$|2x + y - 3| < \varepsilon$$

as desired. ■

Quiz 8 Name:________________________

Directions: Show all of your work and justify all of your answers.

For each of the following, find all local extrema and saddle points.

(1) 1. $f(x,y) = x^3 + xy^2 + 2x^2 + y^2$

$f_x(x,y) = 3x^2 + y^2 + 4x$

$f_y(x,y) = 2xy + 2y = 2y(x+1)$

$f_{xx}(x,y) = 6x + 4$

$f_{yy}(x,y) = 2y + 2$

$f_{xy}(x,y) = 2y$

$f_y(x,0) = 0$

$f_y(\text{-}1,y) = 0$

$f_x(x,0) = 3x^2 + 4x = x(3x+4)$

$f_x(\text{-}1,y) = y^2 - 1$

Critical points: (0,0), ($\text{-}\frac{4}{3}$,0), (-1,-1), (-1,1)

$D(0,0) = 8$

$f_{xx}(0,0) = 4$

Local min: 0 at (0,0)

$D(\text{-}\frac{4}{3},0) = \frac{8}{3}$

$f_{xx}\left(\text{-}\frac{4}{3},0\right) = \text{-}4$

Local max: $\frac{160}{27}$ at $\left(\text{-}\frac{4}{3},0\right)$

$D(\text{-}1,\text{-}1) = \text{-}4$

Saddle point: (-1,-1,1)

$D(\text{-}1,1) = \text{-}12$

Saddle point: (-1,1,1)

(1) 2. $f(x,y) = x^2y + 4xy^2 - xy$

$f_x(x,y) = 2xy + 4y^2 - y$

$f_y(x,y) = x^2 + 8xy - x$

$f_{xx}(x,y) = 2y$

$f_{yy}(x,y) = 8x$

$f_{xy}(x,y) = 2x + 8y - 1$

$f_x(x,y) = y(2x + 4y - 1)$

$y = 0$ or $y = \text{-}\frac{1}{2}x + \frac{1}{4}$

$f_y(x,y) = x(x + 8y - 1)$

$x = 0$ or $y = \text{-}\frac{1}{8}x + \frac{1}{8}$

$\text{-}\frac{1}{2}x + \frac{1}{4} = \text{-}\frac{1}{8}x + \frac{1}{8}$

$\text{-}4x + 2 = \text{-}x + 1$

$x = \frac{1}{3}$

$y = \frac{1}{12}$

Critical points: (0,0), (0,$\frac{1}{4}$), (1,0), ($\frac{1}{3}$,$\frac{1}{12}$)

$D(0,0) = \text{-}1$

Saddle point: (0,0)

$D(0,\frac{1}{4}) = \text{-}1$

Saddle point: (0,$\frac{1}{4}$)

$D(1,0) = \text{-}1$

Saddle point: (1,0)

$D(\frac{1}{3},\frac{1}{12}) = \frac{1}{3}$

$f_{xx}(\frac{1}{3},\frac{1}{12}) = \frac{1}{6}$

Local min: $\text{-}\frac{1}{108}$ at ($\frac{1}{3}$,$\frac{1}{12}$)

Quiz 9 Name:________________________

<u>Directions:</u> Show all of your work and justify all of your answers.

(2) 1. Find the maximum and minimum values of $f(x,y) = e^x + y^2$ on the region bounded by the triangle with vertices (0,0), (1,0), and (1,1).

Since $f_x(x,y) = e^x \geq 0$ for all $x \in \mathbb{R}$, f has no critical points. Since $g(x) = e^x$ and $h(y) = y^2$ are increasing on $[0,\infty)$, f has a local minimum of 1 at (0,0) and a local maximum of $e + 1$ at (1,1).

Quiz 10 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $f(x,y,z) = x^2y + e^{x+z}$.

(1) a. Find the gradient of f at (1,2,-1).

$f_x(x,y,z) = 2xy + e^{x+z}$

$f_y(x,y,z) = x^2$

$f_z(x,y,z) = e^{x+z}$

$\nabla f(1,2,\text{-}1) = \langle 5,1,1 \rangle$

(1) b. Find the directional derivative of f at (1,2,-1) in the direction of $\left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle$.

$\nabla f(1,2,\text{-}1) \cdot \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle = \frac{5\sqrt{3}}{3} + \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = \frac{7\sqrt{3}}{3}$

(1) c. In what direction does the maximum rate of change of f occur at (1,2,-1)?

$\nabla f(1,2,\text{-}1) = \langle 5,1,1 \rangle$

(1) d. What is the maximum rate of change of f at (1,2,-1)?

$\|\nabla f(1,2,\text{-}1)\| = \sqrt{27} = 3\sqrt{3}$

Quiz 11 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Maximize and minimize $f(x,y,z) = x^2 + y + yz$ subject to $x^2 + y^2 + z^2 = 1$ and $y + z = 0$.

$g(x,y,z) = x^2 + y^2 + z^2 - 1$ $\nabla g(x,y,z) = \langle 2x,2y,2z \rangle$

$h(x,y,z) = y + z$ $\nabla h(x,y,z) = \langle 0,1,1 \rangle$

$\nabla f(x,y,z) = \langle 2x,z+1,y \rangle$ $\langle 2x,z+1,y \rangle = \lambda \langle 2x,2y,2z \rangle + \mu \langle 0,1,1 \rangle$

$2x = 2\lambda x$ $z + 1 = 2\lambda y + \mu$ $y = 2\lambda z + \mu$

From the first equation, $x = 0$ or $\lambda = 1$.

Note that $y = -z$ from the second constraint equation.

So if $x = 0$, then either $y = -\frac{\sqrt{2}}{2}$ and $z = \frac{\sqrt{2}}{2}$ or $y = \frac{\sqrt{2}}{2}$ and $z = -\frac{\sqrt{2}}{2}$.

If $\lambda = 1$, then the second and third equations above yield the following.

$z + 1 = 2\lambda y + \mu$ $y = 2\lambda z + \mu$

$z + 1 = 2y + \mu$ $y = 2z + \mu$

$z + 1 = -2z + \mu$ $-z = 2z + \mu$

$3z = \mu - 1$ $3z = -\mu$

So $\mu - 1 = -\mu$ which means that $\mu = \frac{1}{2}$. In this case $z = -\frac{1}{6}$, $y = \frac{1}{6}$, and $x = \pm\sqrt{\frac{17}{18}}$.

$f\left(0,-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}+1}{2}$ (min) $f\left(-\sqrt{\frac{17}{18}},\frac{1}{6},-\frac{1}{6}\right) = \frac{13}{12}$ (max)

$f\left(0,\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}-1}{2}$ $f\left(\sqrt{\frac{17}{18}},\frac{1}{6},-\frac{1}{6}\right) = \frac{13}{12}$ (max)

Quiz 12 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $R = \{(x,y) : 1 \leq x \leq 3, \text{-}2 \leq y \leq 1\}$ and calculate $\iint\limits_R \big(3x^2y - 4xy + 1\big)dA$.

$$\iint\limits_R \big(3x^2y - 4xy + 1\big)dA$$

$$= \int_{-2}^{1}\int_{1}^{3}\big(3x^2y - 4xy + 1\big)dx\,dy$$

$$= \int_{-2}^{1}\big(x^3y - 2x^2y + x\big)\Big|_1^3\,dy$$

$$= \int_{-2}^{1}\big(27y - 18y + 3 - (y - 2y + 1)\big)dy$$

$$= \int_{-2}^{1}\big(10y + 2\big)dy$$

$$= \big(5y^2 + 2y\big)\Big|_{-2}^{1}$$

$$= 5 + 2 - (20 - 4)$$

$$= -9$$

Quiz 13 Name:________________________

Directions: Show all of your work and justify all of your answers.

(Page 1020: 49) **(2) 1.** Evaluate the following integral. Hint: Reverse the order of integration.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$$

Quiz 14 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let $R = \{(x,y,z) : x^2 + y^2 \leq 1, 0 \leq z \leq 3\}$ and calculate $\displaystyle\iiint\limits_R ze^{x^2+y^2+z^2}\,dV$.

$$\iiint\limits_R ze^{x^2+y^2+z^2}\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^3 rze^{r^2+z^2}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \tfrac{1}{2}re^{r^2+z^2}\Big|_0^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\tfrac{1}{2}re^{r^2+9} - \tfrac{1}{2}re^{r^2}\right)dr\,d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{1}{4}e^{r^2+9} - \tfrac{1}{4}e^{r^2}\right)\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{1}{4}e^{10} - \tfrac{1}{4}e - \left(\tfrac{1}{4}e^9 - \tfrac{1}{4}\right)\right]d\theta$$

$$= \tfrac{1}{4}\int_0^{2\pi}\left(e^{10} - e^9 - e + 1\right)d\theta$$

$$= \tfrac{1}{4}\theta\left(e^{10} - e^9 - e + 1\right)\Big|_0^{2\pi}$$

$$= \tfrac{\pi}{2}\left(e^{10} - e^9 - e + 1\right)$$

Quiz 15 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $R = \left\{(x,y) : -2 \leq x \leq 2, -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\right\}$ and calculate $\displaystyle\iint_R 2e^{x^2+y^2}\,dA$.

Note that R is the circle centered at (0,0) with radius 2. Use polar coordinates.

$$\iint_R 2e^{x^2+y^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 2re^{r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} e^{r^2}\Big|_0^2\,d\theta$$

$$= \int_0^{2\pi}\left(e^4-1\right)d\theta$$

$$= \theta\left(e^4-1\right)\Big|_0^{2\pi}$$

$$= 2\pi\left(e^4-1\right)$$

Quiz 16 Name:_________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let R be the unit sphere and calculate $\displaystyle\iiint\limits_R \frac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}\,dV$.

$$\iiint\limits_R \frac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}\,dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \frac{\rho^2\sin(\phi)\sin(\rho^2)}{\rho}\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho\sin(\phi)\sin(\rho^2)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} -\tfrac{1}{2}\cos(\rho^2)\sin(\phi)\Big|_0^1\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \left(-\tfrac{1}{2}\cos 1+\tfrac{1}{2}\right)\sin(\phi)\,d\theta\,d\phi$$

$$= \int_0^{\pi} \theta\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 1\right)\sin(\phi)\Big|_0^{2\pi}\,d\phi$$

$$= \int_0^{\pi} 2\pi\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 1\right)\sin(\phi)\,d\phi$$

$$= -2\pi\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 1\right)\cos(\phi)\Big|_0^{\pi}$$

$$= 4\pi\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 1\right)$$

$$= 2\pi\left(1-\cos 1\right)$$

Due 3:30 Friday May 1, 2015

Directions:

1. Show all of your work.

2. Write on only one side of the paper.

3. Use separate paper for each problem.

4. Put the problems in numerical order.

5. Staple your pages.

(1) 1. Page 1096: 9

(1) 2. Page 1107: 17

(1) 3. Page 1114: 11

(1) 4. Page 1121: 9

(1) 5. Page 1121: 15

(1) 6. Page 1132: 49

(1) 7. Page 1144: 7

(1) 8. Page 1144: 21

Section 2.2: Exam 1

Exam 1 Math 2673 Spring 2015

Name:____________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

9. Let $\boldsymbol{v} = \langle 0,\text{-}1,2\rangle$ and $\boldsymbol{w} = \langle 1,\text{-}1,3\rangle$.

(10) a. Find $\cos\theta$ where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{7}{\sqrt{5}\cdot\sqrt{11}} = \frac{7}{\sqrt{55}}$

(10) b. Find $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$.

$\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v} = \frac{7}{5}\langle 0,\text{-}1,2\rangle$

(10) c. Give the equations of the lines through the point (1,1,1) in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$.

$x = 1$ $\qquad\qquad$ $x = 1 + t$

$y = 1 - s$ $\qquad\qquad$ $y = 1 - t$

$z = 1 + 2s$ $\qquad\qquad$ $z = 1 + 3t$

(10) d. Give the equation of the plane containing the lines from the previous part. You need not to have done the previous part to do this part.

$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle -1,2,1 \rangle$

$-(x-1) + 2(y-1) + (z-1) = 0$

(10) 10. Give the equation of the sphere with center (-1,2,6) and radius 3.

$(x+1)^2 + (y-2)^2 + (z-6)^2 = 9$

(Page 831: 51) **(10) 11.** A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

(10) 12. Find the volume of the parallelepiped determined by $\langle -1,0,4 \rangle$, $\langle 0,2,3 \rangle$, and $\langle -1,3,1 \rangle$.

$$|\langle -1,0,4 \rangle \cdot (\langle 0,2,3 \rangle \times \langle -1,3,1 \rangle)| = \left\| \begin{matrix} -1 & 0 & 4 \\ 0 & 2 & 3 \\ -1 & 3 & 1 \end{matrix} \right\| = |(-1)(-7) - (0)(3) + (4)(2)| = 15$$

13. Identify and describe each of the following.

(10) a. $(x+1)^2 + (y-3)^2 = 4,\ 0 \leq z \leq 3$

This is a right circular cylinder. The base is the circle in the xy-plane centered at (-1,3) with radius 2. The height of the cylinder is 3.

(10) b. $z = \frac{x^2}{4} + \frac{y^2}{9}$

Elliptic paraboloid.

(10) c. $(x+1)^2 + \frac{(y+4)^2}{16} + z^2 = 1$

Ellipsoid.

14. Let $\boldsymbol{f}(t) = \langle \sin t, \cos t, t \rangle$

(10) a. Sketch the graph of $\boldsymbol{f}$.

(10) b. Give the equation of the line that is tangent to the graph of $\boldsymbol{f}$ at the point $\left(1,0,\frac{\pi}{2}\right)$.

$\boldsymbol{f}'(t) = \langle \cos t, \text{-} \sin t, 1\rangle$

$\boldsymbol{f}'\left(\frac{\pi}{2}\right) = \langle 0, \text{-}1, 1\rangle$

$x = 1$

$y = \text{-}t$

$z = 1 + t$

Section 2.3: Exam 2

Exam 2 Math 2673 Spring 2015

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

15. Let $\boldsymbol{r}(t) = \left\langle t, \frac{2\sqrt{2}}{3} t^{\frac{3}{2}}, \frac{1}{2}t^2 \right\rangle$.

(10) a. Calculate $\boldsymbol{r}'(t)$ and $\|\boldsymbol{r}'(t)\|$.

$\boldsymbol{r}'(t) = \left\langle 1, \sqrt{2} t^{\frac{1}{2}}, t \right\rangle = \langle 1, \sqrt{2t}, t\rangle$

$\|\boldsymbol{r}'(t)\| = \sqrt{1 + 2t + t^2} = \sqrt{(t+1)^2} = |t+1|$

(10) b. Find the arc length of the curve defined by $\boldsymbol{r}(t)$ as t ranges from 0 to 1.

$$\int_0^1 |t+1|\, dt = \int_0^1 \big(t+1\big) dt = \left(\tfrac{1}{2}t^2 + t\right)\Big|_0^1 = \tfrac{3}{2}$$

(10) c. Calculate the unit tangent vector at the point $t = 2$.

$$\boldsymbol{T}(t) = \frac{r'(t)}{\|r'(t)\|} = \tfrac{1}{|t+1|} \langle 1, \sqrt{2t}, t\rangle$$

$\boldsymbol{T}(2) = \frac{1}{3} \langle 1, 2, 2\rangle$

(10) 16. Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

$\boldsymbol{a}(t) = \boldsymbol{i} + 2\boldsymbol{j}$ $\qquad$ $\boldsymbol{v}(0) = \boldsymbol{k}$ $\qquad$ $\boldsymbol{r}(0) = \boldsymbol{i}$

$\boldsymbol{v}(t) = t\boldsymbol{i} + 2t\boldsymbol{j} + \boldsymbol{k} = \langle t, 2t, 1\rangle$

$\boldsymbol{r}(t) = \frac{1}{2}t^2\boldsymbol{i} + t^2\boldsymbol{j} + t\boldsymbol{k} + \boldsymbol{i} = \left\langle \frac{1}{2}t^2 + 1, t^2, t\right\rangle$

17. Let $f(x,y) = \sqrt{x^2 + y^2}$.

(10) a. Give the domain and range of f.

Domain: $\mathbb{R}^2$

Range: $[0,\infty)$

(10) b. Sketch or describe the graph of f.

18. Calculate the following limits.

(10) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x + y}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x+y}$$

$$= \lim_{(x,y)\to(0,0)} \left(x + y\right)$$

$$= 0$$

(10) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y}{x + y^2}$ DNE

$$\lim_{(x,0)\to(0,0)} \frac{x^2 + y}{x + y^2}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x^2}{x}$$

$$= \lim_{(x,0)\to(0,0)} x$$

$$= 0$$

$$= \lim_{(0,y)\to(0,0)} \frac{x^2 + y}{x + y^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{y}{y^2}$$

$$= \lim_{(0,y)\to(0,0)} \frac{1}{y}$$

DNE

19. Let $f(x,y) = x^2y^2 + 2xy$.

$$z = \sqrt{x^2 + y^2}$$

$$z^2 = x^2 + y^2$$

Top half of a cone.

(10) a. Calculate all first order partial derivatives and all second order partial derivatives of f.

$$f_x(x,y) = 2xy^2 + 2y$$

$$f_y(x,y) = 2x^2y + 2x$$

$$f_{xx}(x,y) = 2y^2$$

$$f_{yy}(x,y) = 2x^2$$

$$f_{xy}(x,y) = 4xy + 2$$

$$f_{yx}(x,y) = 4xy + 2$$

(10) b. Find the critical points of f.

$$f_x(x,y) = 0$$

$$2xy^2 + 2y = 0$$

$$2y(xy + 1) = 0$$

$$f_y(x,y) = 0$$

$$2x^2y + 2x = 0$$

$$2x(xy + 1) = 0$$

CP: (0,0), all points on the curve $y = -\frac{1}{x}$

(10) c. Find the equation of the plane that is tangent to the graph of f at the point $(2, 1, 8)$.

$$f_x(2,1) = 6$$

$$f_y(2,1) = 12$$

$$z - 8 = 6(x - 2) + 12(y - 1)$$

Section 2.4: Exam 3

Exam 3 Math 2673 Spring 2015

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

20. Let $f(x,y,z) = xe^y + x^3 z$.

(10) a. Calculate $\nabla f(x,y,z)$.

$\nabla f(x,y,z) = \langle e^y + 3x^2 z, xe^y, x^3 \rangle$

(10) b. Calculate the directional derivative of f at the point (1,0,2) in the direction of $\frac{1}{\sqrt{3}}\langle 1, \text{-}1, 1\rangle$.

$\nabla f(1,0,2) = \langle 7, 1, 1\rangle$

$\langle 7, 1, 1\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, \text{-}1, 1\rangle = \frac{1}{\sqrt{3}}(7 - 1 + 1) = \frac{7}{\sqrt{3}}$

(10) 21. Let $z = x^2 y + x^4$ where $x = t^3$ and $y = \sin t$. Calculate $\frac{dz}{dt}$.

$\frac{\partial z}{\partial x} = 2xy + 4x^3$

$\frac{\partial z}{\partial y} = x^2$

$\frac{dx}{dt} = 3t^2$

$\frac{dy}{dt} = \cos t$

$\frac{dz}{dt} = (2xy + 4x^3)(3t^2) + x^2(\cos t) = [2(t^3)(\cos t) + 4t^9](3t^2) + t^6(\cos t) = t^6 \cos t + 6t^5 \cos t + 12t^{11}$

(10) 22. Let $z = e^{xy}$ where $x = t + s^2$ and $y = \sqrt{t}$. Calculate $\frac{\partial z}{\partial s}$.

$\frac{\partial z}{\partial x} = ye^{xy}$

$\frac{\partial z}{\partial y} = xe^{xy}$

$\frac{\partial x}{\partial s} = 2s$

$\frac{\partial y}{\partial s} = 0$

$\frac{\partial z}{\partial s} = ye^{xy}(2s) + xe^{xy}(0) = 2s\sqrt{t}e^{\sqrt{t}(t+s^2)}$

(10) 23. Maximize and minimize $f(x,y,z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$.

(10) 24. Maximize and minimize $f(x,y,z) = -8x + 3y + z$ subject to $x^2 + y = 5$ and $y + z = 3$.

(10) 25. Let $R = [0,2] \times [1,3]$ and calculate $\displaystyle\iint\limits_R (x^2y + x^3)\,dA$.

(10) 26. Let R be the region bounded by the triangle with vertices (0,0), (1,0), and (1,1) and calculate $\iint\limits_R \left(e^{x^2}\right) dA$.

(10) 27. Let T be the solid tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,4) and calculate $\displaystyle\iint_T \tfrac{1}{4}\, dV$.

Chapter 3: Spring 2017

Section 3.1: Quizzes

Quiz 17 Name:_______________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the equation of the sphere with center (-1,3,4) and radius 2.

$(x+1)^2 + (y-3)^2 + (z-4)^2 = 4$

(1) 2. Find and simplify an equation for the set of all points in $\mathbb{R}^3$ that are equidistant from the points $(1, 1, 1)$ and $(2, 2, 2)$.

$\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2} = \sqrt{(x-2)^2 + (y-2)^2 + (z-2)^2}$

$(x-1)^2 + (y-1)^2 + (z-1)^2 = (x-2)^2 + (y-2)^2 + (z-2)^2$

$x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 - 2z + 1 = x^2 - 4x + 4 + y^2 - 4y + 4 + z^2 - 4z + 4$

$2x + 2y + 2z = 9$

Quiz 18 Name:_________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that vector $\boldsymbol{v}$ is parallel to vector $\boldsymbol{w} = \langle 1,2,2 \rangle$ and $\|\boldsymbol{v}\| = 5$. Find $\boldsymbol{v}$.

Note that $\|\boldsymbol{w}\| = 3$. So the vector in the direction of $\boldsymbol{w}$ with magnitude 5 is $\frac{5}{3}\boldsymbol{w} = \left\langle \frac{5}{3}, \frac{10}{3}, \frac{10}{3} \right\rangle$

(1) 2. Answer the following as true or false and justify your answer.

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vectors in $\mathbb{R}^3$ such that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$, then one of $\boldsymbol{v}$ or $\boldsymbol{w}$ is $\mathbf{0}$.

False.

Consider $\boldsymbol{i}$ and $\boldsymbol{j}$ for example.

Also, recall that two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

Quiz 19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $\boldsymbol{v} = \langle 1,\text{-}1,2 \rangle$ and $\boldsymbol{w} = \langle 1,0,3 \rangle$. Find $\cos\theta$ where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{||\boldsymbol{v}||||\boldsymbol{w}||} = \frac{7}{\sqrt{6}\cdot\sqrt{10}} = \frac{7}{2\sqrt{15}}$

(Page 831: 51) **(1) 2.** A sled is pulled along a level path through snow by a rope. A 30-lb force acting at an angle of 40° above the horizontal moves the sled 80 ft. Find the work done by the force.

Quiz 20 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find the equation of the plane that contains the points $(1, 1, 4)$, $(\text{-}1, \text{-}1, 2)$, and $(2, 0, 1)$.

$\boldsymbol{v} = \langle 1 - \text{-}1, 1 - \text{-}1, 4 - 2 \rangle = \langle 2,2,2 \rangle = 2\,\langle 1,1,1 \rangle$

$\boldsymbol{w} = \langle 2 - \text{-}1, 0 - \text{-}1, 1 - 2 \rangle = \langle 3,1,\text{-}1 \rangle$

$\boldsymbol{n} = \langle 1,1,1 \rangle \times \langle 3,1,\text{-}1 \rangle = \langle \text{-}2,4,\text{-}2 \rangle = \text{-}2\,\langle 1,\text{-}2,1 \rangle$

$(x - 1) - 2(y - 1) + (z - 4) = 0$

$x - 2y + z = 3$

(2) 2. Give the equation of the line of intersection of the following planes.

$2x - y - z = 2$

$y - z = 0$

$\boldsymbol{n}_1 = \langle 2,\text{-}1,\text{-}1 \rangle$

$\boldsymbol{n}_2 = \langle 0,1,\text{-}1 \rangle$

$\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle 2,2,2 \rangle = 2\,\langle 1,1,1 \rangle$

Point on both planes: (1,0,0)

$x = 1 + t$

$y = t$

$z = t$

Quiz 21 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Sketch or describe the curve of the following vector function.

$\boldsymbol{r}(t) = \langle \sin t, e^t, \cos t \rangle$

Let C be the cylinder with base $x^2 + z^2 = 1$ (circle centered at (0,0) with radius 1) in the xz-plane. As $t \to \infty$, the curve spirals around C in the direction of the positive y-axis. Note that the y-coordinate increases exponentially. As $t \to -\infty$, the curve spirals around C toward (but never touching) the xz-plane.

(2) 2. Give the vector equation of the curve where the following surfaces intersect.

$x^2 + y^2 = z$

$x^2 - y^2 = 1$

Adding the two equations together, yields

$z + 1 = 2x^2$

$z = 2x^2 - 1$

From the first equation, we have

$y = \sqrt{x^2 - 1}$

$\boldsymbol{r}(t) = \left\langle t, \sqrt{t^2 - 1}, 2t^2 - 1 \right\rangle$

Quiz 22 Name:____________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{r}(t) = \langle \sin(\pi t), e^{t-1}, t \ln t \rangle$.

(1) a. Find the derivative of $\boldsymbol{r}$.

$\boldsymbol{r}'(t) = \langle \pi \cos(\pi t), e^{t-1}, \ln t + 1 \rangle$

(2) b. Find the equation of the line that is tangent to the graph of $\boldsymbol{r}$ where $t = 1$.

$\boldsymbol{r}(1) = \langle 0,1,0 \rangle$

$\boldsymbol{r}'(1) = \langle -\pi,1,1 \rangle$

$\boldsymbol{f}(s) = \langle 0,1,0 \rangle + s \langle -\pi,1,1 \rangle$

Quiz 23 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3} t^{\frac{3}{2}}, 2t \right\rangle$.

(1) a. Calculate each of the following.

$$\boldsymbol{r}'(t) = \left\langle 2t, 2\sqrt{2}t^{\frac{1}{2}}, 2 \right\rangle = 2\left\langle t, \sqrt{2t}, 1 \right\rangle$$

$$\|\boldsymbol{r}'(t)\| = 2\sqrt{t^2+2t+1} = 2\sqrt{(t+1)^2} = 2|t+1|$$

$$\boldsymbol{r}''(t) = \left\langle 2, \sqrt{2}t^{-\frac{1}{2}}, 0 \right\rangle$$

(1) b. Find the arc length of the curve defined by $\boldsymbol{r}(t)$ as t ranges from 0 to 1.

$$\int_0^1 2|t+1|\,dt = \int_0^1 2(t+1)\,dt = \int_0^1 \left(2t+2\right)dt = \left(t^2+2t\right)\Big|_0^1 = 3$$

(1) c. Calculate the curvature at the point $t = 2$.

$$\kappa(2) = \frac{\|\boldsymbol{r}'(2) \times \boldsymbol{r}''(2)\|}{\|\boldsymbol{r}'(2)\|^3} = \frac{\|\langle 4,4,2\rangle \times \langle 2,1,0\rangle\|}{6^3} = \frac{\|\langle -2,4,-4\rangle\|}{6^3} = \frac{6}{6^3} = \frac{1}{36}$$

Quiz 24 Name:_________________________

Directions: Show all of your work and justify all of your answers.

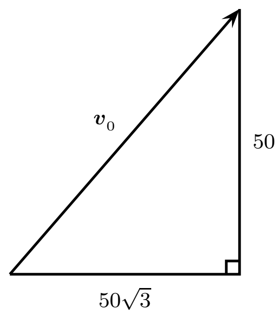
(1) 1. Let $\boldsymbol{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3} t^{\frac{3}{2}}, 2t \right\rangle$. Calculate the curvature at the point $t = 2$.

Quiz 25 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. A projectile is launched with an initial speed of 100 feet per second at an angle of $\frac{\pi}{6}$ to the horizontal. Assume that the only force acting on the object is gravity.)

a. Find the initial velocity vector.



$\boldsymbol{v}_0 = 50\sqrt{3}\boldsymbol{i} + 50\boldsymbol{j}$

b. Find the vector function that describes velocity.

$\boldsymbol{a}(t) = \text{-}32\boldsymbol{j}$

$\boldsymbol{v}(t) = \text{-}32t\boldsymbol{j} + \boldsymbol{v}(0)$

$\boldsymbol{v}(t) = \text{-}32t\boldsymbol{j} + 50\sqrt{3}\boldsymbol{i} + 50\boldsymbol{j}$

$\boldsymbol{v}(t) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32t + 50)\boldsymbol{j}$

c. Find the vector function that describes motion.

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j} + \boldsymbol{s}(0)$

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j}$

d. Find the maximum height.

$\boldsymbol{v}(t) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32t + 50)\boldsymbol{j}$

$\text{-}32t + 50 = 0$

$t = \frac{25}{16}$

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j}$

$\text{-}16\left(\frac{25}{16}\right)^2 + 50\left(\frac{25}{16}\right) = \frac{625}{16}$

$\frac{625}{16}$ ft

e. Find the horizontal range.

$\boldsymbol{s}(t) = 50\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 50t)\boldsymbol{j}$

$\text{-}16t^2 + 50t = 0$

$t(\text{-}16t + 50) = 0$

$t = \frac{25}{8}$

$50\sqrt{3} \cdot \frac{25}{8} = \frac{625\sqrt{3}}{4}$

f. Find the speed of impact.

$\boldsymbol{v}(t) = 50\sqrt{3}\boldsymbol{i} + (\text{-}32t + 50)\boldsymbol{j}$

$\boldsymbol{v}\left(\frac{25}{8}\right) = 50\sqrt{3}\boldsymbol{i} + \left(\text{-}32\left(\frac{25}{8}\right) + 50\right)\boldsymbol{j}$

$\left\|\boldsymbol{v}\left(\frac{25}{8}\right)\right\| = 100$

100 ft/sec.

Quiz 26 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. For the following, find all local extrema and saddle points.

$f(x,y) = \frac{2}{3}x^3 + xy^2 + 3x^2 + y^2$

$f_x(x,y) = 2x^2 + y^2 + 6x$

$f_y(x,y) = 2xy + 2y = 2y(x+1)$

$f_{xx}(x,y) = 4x + 6$

$f_{yy}(x,y) = 2x + 2$

$f_{xy}(x,y) = 2y$

$D(x,y) = (4x+6)(2x+2) - 4y^2$

Solve:

$f_y(x,y) = 0$

$2y(x+1) = 0$

$x = -1$ or $y = 0$

Solve:

$f_x(x,0) = 0$

$2x^2 + 6x = 0$

$2x(x+3) = 0$

$x = 0$ or $x = -3$

Solve:

$f_x(-1,y) = 0$

$y^2 - 4 = 0$

$(y+2)(y-2) = 0$

$y = -2$ or $y = 2$

Critical points: (0,0), (-3,0), (-1,-2), (-1,2)

$D(0,0) = 12$

$f_{xx}(0,0) = 6$

Local min: 0 at (0,0)

$D(-3,0) = 24$

$f_{xx}(-3,0) = -6$

Local max: 9 at (-3,0)

$D(-1,-2) = -16$

Saddle point: $\left(-1,-2,\frac{7}{3}\right)$

$D(-1,2) = -16$

Saddle point: $\left(-1,2,\frac{7}{3}\right)$

Quiz 27 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find the absolute extrema of the function $f(x,y) = x^2 e^y$ on the region in $\mathbb{R}^2$ bounded by the unit circle.

$f_x(x,y) = 2xe^y$

$f_y(x,y) = x^2 e^y$

Critical Points: $(0,y)$ for all $y \in [-1,1]$

$f(0,y) = 0$

Boundary Points: All points on the unit circle.

$x^2 + y^2 = 1$

$x = \sqrt{1-y^2}$

$f(\sqrt{1-y^2},y) = (1-y^2)e^y$

Consider

$g(y) = (1-y^2)e^y$ on $[-1,1]$

$g'(y) = -2ye^y + (1-y^2)e^y = -e^y(y^2 + 2y - 1)$

To find the critical points of g, solve:

$y^2 + 2y - 1 = 0$

$y = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Note that $-1 - \sqrt{2} \notin [-1,1]$. So $y = \sqrt{2} - 1$.

$y^2 = \left(\sqrt{2}-1\right)^2 = 3 - 2\sqrt{2}$

$x^2 = 1 - y^2 = 1 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$

$x = \pm\sqrt{2(\sqrt{2}-1)}$

Absolute min: 0 at $(0,y)$ for all $y \in [-1,1]$

Absolute max: $2(\sqrt{2}-1)e^{\sqrt{2}-1}$ at $\left(-\sqrt{2(\sqrt{2}-1)}, \sqrt{2}-1\right)$ and $\left(\sqrt{2(\sqrt{2}-1)}, \sqrt{2}-1\right)$

Quiz 28 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Find the linearization of $f(x,y) = 2x^3 - xy + y^2$; at the point (1,-1,4). Use it to approximate $f(.99,\text{-}1.01)$.

$f_x(x,y) = 6x^2 - y$

$f_y(x,y) = \text{-}x + 2y$

$f_x(1,\text{-}1) = 7$

$f_y(1,\text{-}1) = \text{-}3$

Linearization: $L(x,y) = 7(x-1) - 3(y+1) + 4$

$f(.99,\text{-}1.01) \approx L(.99,\text{-}1.01) = 7(.99-1) - 3(\text{-}1.01+1) + 4 = \text{-}.07 + .03 + 4 = 3.96$

Quiz 29 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Find the maximum value of the function $f(x,y) = x^2y$ subject to $x^2 + y = 1$.

Let $g(x,y) = x^2 + y - 1$.

$\nabla f(x,y) = \langle 2xy, x^2 \rangle$

$\nabla g(x,y) = \langle 2x, 1 \rangle$

$2xy = \lambda 2x$

$x = 0$ or $\lambda = y$

$f(0,y) = 0$

Suppose $\lambda = y$.

Then $x^2 = \lambda = y$.

$x^2 + y = 1$

$2y = 1$

$y = \frac{1}{2}$

$y = \lambda = x^2 = \frac{1}{2}$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$f\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\right) = \frac{1}{4}$ (max)

$f\left(-\sqrt{\frac{1}{2}}, \frac{1}{2}\right) = \frac{1}{4}$ (max)

Explain why no such minimum exists.

By letting y be a negative number with a large absolute value, x^2y can be made arbitrarily small. Below is a more precise explanation.

Suppose that $M \leq -1$. Choose $x \in \mathbb{R}$ such that $-x^2 < M$ and let $y = 1 - x^2$. Then $x^2 + y = 1$ and $f(x,y) = x^2y < M$.

Quiz 30 Name:_______________________

Directions: Show all of your work and justify all of your answers.

For each of the following, give the coordinates of the point in the the other two coordinate systems.

(1) 1. Rectangular coordinates: $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 0\right)$

$r^2 = x^2 + y^2 = 9$

$r = 3$

$\rho^2 = x^2 + y^2 + z^2 = 9$

$\rho = 3$

$x = r\cos\theta$

$\frac{3}{2} = 3\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$\rho = r\sin\phi$

$3 = 3\sin\phi$

$\sin\phi = 1$

$\phi = \frac{\pi}{2}$

Cylindrical: $\left(3, \frac{\pi}{3}, 0\right)$

Spherical: $\left(3, \frac{\pi}{3}, \frac{\pi}{2}\right)$

(1) 2. Cylindrical coordinates: $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{6}, \frac{1}{2}\right)$

$x = r\cos\theta = \frac{\sqrt{3}}{2}\cos\frac{\pi}{6} = \frac{3}{4}$

$y = r\sin\theta = \frac{\sqrt{3}}{2}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{4}$

$\rho^2 = r^2 + z^2 = \frac{3}{4} + \frac{1}{4} = 1$

$\rho = 1$

$r = \rho\sin\phi$

$\frac{\sqrt{3}}{2} = \sin\phi$

$\phi = \frac{\pi}{3}$

Rectangular: $\left(\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$

Spherical: $\left(1, \frac{\pi}{6}, \frac{\pi}{3}\right)$

(1) 3. Spherical cooridinates: $\left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)$

$\rho = 4$

$\theta = \frac{\pi}{3}$

$\phi = \frac{\pi}{6}$

$r = 4\sin\frac{\pi}{6} = 2$

$z = 4\cos\frac{\pi}{6} = 2\sqrt{3}$

$x = 2\cos\frac{\pi}{3} = 1$

$y = 2\sin\frac{\pi}{3} = \sqrt{3}$

Rectangular: $\left(1, \sqrt{3}, 2\sqrt{3}\right)$

Cylindrical: $\left(2, \frac{\pi}{3}, 2\sqrt{3}\right)$

Quiz 31 Name:________________________

<u>Directions:</u> Show all of your work and justify all of your answers.

1. For the following, T is the solid tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,1), $R = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1, 0 \leq x, 0 \leq z \leq 3\}$, and S is the region of $\mathbb{R}^3$ that lies above the the top half of the cone $\frac{z^2}{3} = x^2+y^2$ and below the sphere $x^2 + y^2 + (z - \sqrt{3})^2 = 1$. Evaluate the following.

(1) a. $\displaystyle\iiint_T x^2 y \, dV$ **(1) b.** $\displaystyle\iiint_R e^{x^2+y^2} \, dV$ **(1) c.** $\displaystyle\iiint_S 1 \, dV$

Section 3.2: Exam 1

Exam 1 Math 2673 Spring 2017

(10) 2. Give the equation of the sphere with center (1,-1,2) and containing the point (2,1,0).

$r = \sqrt{(1-2)^2 + (-1-1)^2 + (2-0)^2} = 3$

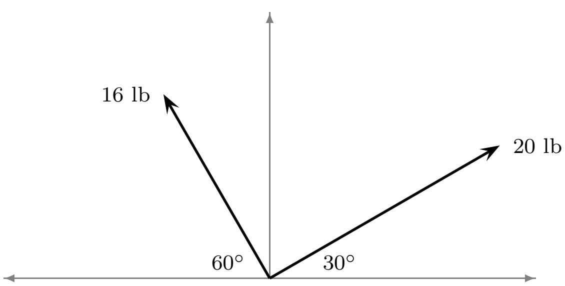
$(x-1)^2 + (y+1)^2 + (z-2)^2 = 9$

(10) 3. Describe the region in $\mathbb{R}^3$ defined by the following inequality.

$1 \le y^2 + z^2 \le 4$

Let C_1 be the cylinder with base $y^2 + z^2 = 1$ (circle centered at (0,0) with radius 1) in the yz-plane and C_2 be the cylinder with base $y^2 + z^2 = 4$ (circle centered at (0,0) with radius 2) in the yz-plane. The region defined by the inequality is the set of points that are either on the surface of C_2 or in the interior of C_2 but are not in the interior of C_1.

(10) 4. Two force vectors are given below. Find the resultant force vector.



$(10\sqrt{3}\boldsymbol{i} + 10\boldsymbol{j}) + (-8\boldsymbol{i} + 8\sqrt{3}\boldsymbol{j}) = (10\sqrt{3} - 8)\boldsymbol{i} + (8\sqrt{3} + 10)\boldsymbol{j} = \left\langle 10\sqrt{3} - 8, 8\sqrt{3} + 10 \right\rangle$

(10) 5. Determine whether the vectors given below are parallel, perpendicular, or neither.

$\langle 1,2,-2 \rangle$ $\qquad$ $\langle 2,3,4 \rangle$

Perpendicular.

$\langle 1,2,-2 \rangle \cdot \langle 2,3,4 \rangle = 2 + 6 - 8 = 0$

6. Let $\boldsymbol{v} = \langle 1,1,2 \rangle$ and $\boldsymbol{w} = \langle 3,1,3 \rangle$.

(10) a. Find $\text{comp}_{\boldsymbol{v}}\boldsymbol{w}$.

$\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{10}{\sqrt{6}}$

(10) b. Find $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$.

$\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \left(\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v} = \frac{5}{3}\langle 1,1,2 \rangle$

(10) 7. Find the direction cosines of $\left\langle -4,2,\sqrt{5} \right\rangle$.

$\|\langle\text{-}4,2,\sqrt{5}\rangle\| = 5$

$\cos\alpha = \frac{-4}{5}$ $\cos\beta = \frac{2}{5}$ $\cos\gamma = \frac{\sqrt{5}}{5}$

(10) 8. Find the volume of the parallelepiped determined by $\langle 1,2,4\rangle$, $\langle 0,\text{-}2,1\rangle$, and $\langle 1,1,3\rangle$.

$$\left\| \begin{vmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 1 & 1 & 3 \end{vmatrix} \right\| = |1(-6-1) - 2(0-1) + 4(0 - -2)| = 3$$

(10) 9. Find the equation of the plane containing the point (1,-1,3) with normal vector $\boldsymbol{n} = \langle 1,\text{-}2,1\rangle$.

$1(x-1) - 2(y+1) + 1(z-3) = 0$

(10) 10. Give the equation of the line containing the points (-1,2,0) and (-4,1,2).

$\boldsymbol{v} = \langle 3,1,\text{-}2\rangle$ $x = \text{-}1 + 3t$ $y = 2 + t$ $z = 0 - 2t$

11. Identify and describe each of the following.

(10) a. $\frac{x}{2} = \frac{y^2}{9} + \frac{z^2}{4}$

This is an elliptic paraboloid whose axis is the positive x-axis.

(10) b. $x^2 - \frac{y^2}{4} + \frac{z^2}{3} = 2$

This is a hyperboloid of one sheet whose axis is the y-axis.

Section 3.3: Exam 2

Exam 2 Math 2673 Spring 2017

Name:________________________

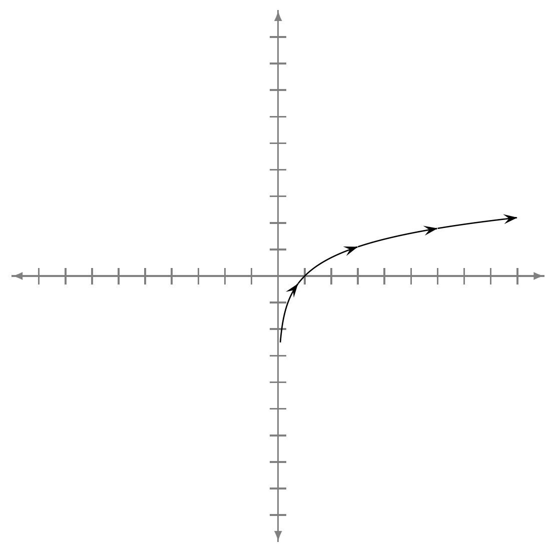
Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

(10) 1. Sketch the curve of the given vector equation. Draw arrows along the curve in the direction of increasing t.

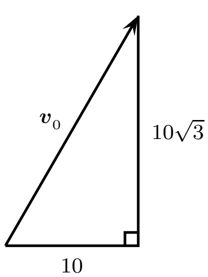
$\boldsymbol{r}(t) = \langle e^t, t\rangle$

$x = e^y$

$y = \ln x$



(10) 2. A projectile is launched due east with an initial speed of 20 feet per second at an angle of $\frac{\pi}{3}$ to the horizontal. Further, suppose that the acceleration due to gravity and wind blowing due south is given by $\boldsymbol{a}(t) = \langle 0,\text{-}20,\text{-}32\rangle$. Give the vector function that represents the projectile's position at time t.



$\boldsymbol{v}_0 = \left\langle 10,0,10\sqrt{3}\right\rangle$

$\boldsymbol{a}(t) = \langle 0,\text{-}20,\text{-}32\rangle$

$\boldsymbol{v}(t)$

$= \langle 0,\text{-}20t,\text{-}32t\rangle + \boldsymbol{v}_0$

$= \left\langle 10,\text{-}20t,\text{-}32t + 10\sqrt{3}\right\rangle$

$\boldsymbol{s}(t)$

$= \left\langle 10t,\text{-}10t^2,\text{-}16t^2 + 10\sqrt{3}t\right\rangle + \boldsymbol{s}_0$

$= \left\langle 10t,\text{-}10t^2,\text{-}16t^2 + 10\sqrt{3}t\right\rangle + \boldsymbol{0}$

$= \left\langle 10t,\text{-}10t^2,\text{-}16t^2 + 10\sqrt{3}t\right\rangle$

3. Let $\boldsymbol{r}(t) = \left\langle \cos t, \sin t, \sqrt{3}t \right\rangle$.

(10) a. Find $\boldsymbol{r}'(t)$.

$\boldsymbol{r}'(t) = \left\langle \text{-} \sin t, \cos t, \sqrt{3} \right\rangle$

(10) b. Find the unit tangent vector $\boldsymbol{T}(t)$.

$\|\boldsymbol{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 3} = 2$

$\boldsymbol{T}(t) = \frac{1}{2}\boldsymbol{r}'(t) = \frac{1}{2}\left\langle \text{-} \sin t, \cos t, \sqrt{3} \right\rangle$

(10) c. Find the unit normal vector $\boldsymbol{N}(t)$.

$\boldsymbol{T}'(t) = \frac{1}{2}\langle \text{-} \cos t, \text{-} \sin t, 0 \rangle$

$\|\boldsymbol{T}'(t)\| = \frac{1}{2}\sqrt{\cos^2 t + \sin^2 t + 0} = \frac{1}{2}$

$\boldsymbol{N}(t) = 2\boldsymbol{T}'(t) = = \langle \text{-} \cos t, \text{-} \sin t, 0 \rangle$

(10) d. Find the binormal vector $\boldsymbol{B}(t)$.

$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t) = \frac{1}{2}\langle \text{-} \cos t, \text{-} \sin t, 0 \rangle \times \langle \text{-} \cos t, \text{-} \sin t, 0 \rangle = \frac{1}{2}\left\langle \sqrt{3} \sin t, \text{-}\sqrt{3} \cos t, 1 \right\rangle$

(10) e. Find the curvature $\kappa(t)$.

$\kappa(t) = \frac{\|\boldsymbol{T}'(t)\|}{\|\boldsymbol{r}'(t)\|} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$

(10) f. Reparameterize the curve with respect to arc length beginning at the point $t = 0$.

$$s = \int_0^t \|\boldsymbol{r}'(u)\|\, du = \int_0^t 2\, du = 2t$$

$t = \frac{1}{2}s$

$\boldsymbol{r}(t) = \boldsymbol{r}\left(\frac{1}{2}s\right) = \left\langle \cos \frac{1}{2}s, \sin \frac{1}{2}s, \frac{\sqrt{3}s}{2} \right\rangle$

(10) g. Give the line that is tangent to $\boldsymbol{r}(t)$ at the point $t = \frac{\pi}{2}$.

$\boldsymbol{r}\left(\frac{\pi}{2}\right) = \left\langle 0, 1, \frac{\sqrt{3}\pi}{2} \right\rangle$

$\boldsymbol{r}'\left(\frac{\pi}{2}\right) = \left\langle \text{-}1, 0, \sqrt{3} \right\rangle$

$\boldsymbol{l}(s) = \left\langle 0, 1, \frac{\sqrt{3}\pi}{2} \right\rangle + s\left\langle \text{-}1, 0, \sqrt{3} \right\rangle$

4. For each of the following, find the limit or prove that it does not exist.

(10) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^4}$

$\displaystyle\lim_{(x,0)\to(0,0)} \frac{x\cdot 0}{x^4+0}$

$= \displaystyle\lim_{(x,0)\to(0,0)} \frac{0}{x^4}$

$= 0$

$\displaystyle\lim_{(x,x)\to(0,0)} \frac{x\cdot x^3}{x^4+x^4}$

$= \displaystyle\lim_{(x,x)\to(0,0)} \frac{x^4}{2x^4}$

$= \frac{1}{2}$

Limit DNE.

(10) b. $\displaystyle\lim_{(x,y)\to(\pi,\pi)} e^{\sin x}\cos y$

$= e^{\sin \pi}\cos \pi$

-1

(10) c. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}$

$= \displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2+1}-1}{x^2+y^2}\cdot\frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1}$

$= \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+1-1}{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}$

$= \displaystyle\lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2+y^2+1}+1}$

$= \frac{1}{2}$

5. Let $f(x,y,z) = x^2e^{xy} + z\sin y$. Calculate each of the following.

(10) a. $f_x(x,y,z) = 2xe^{xy} + x^2ye^{xy}$

(10) b. $f_y(x,y,z) = x^3e^{xy} + z\cos y$

(10) c. $f_z(x,y,z) = \sin y$

(10) d. $f_{xyz}(x,y,z)$

$= f_{zxy}(x,y,z)$

$f_z(x,y,z) = \sin y$

$f_{zx}(x,y,z) = 0$

$f_{zxy}(x,y,z) = 0$

Section 3.4: Exam 3

Exam 3 Math 2673 Spring 2017

(10) 1. For the function below, find all local extrema and saddle points.

$f(x,y) = x^2y + x^2 - y$

$f_x(x,y) = 2xy + 2x = 2x(y+1)$

$f_y(x,y) = x^2 - 1 = (x+1)(x-1)$

$f_{xx}(x,y) = y$

$f_{yy}(x,y) = 0$

$f_{xy}(x,y)\ 2x$

$D(x,y) =$ -$4x^2$

CP: (-1,-1) (1,-1)

D(-1,-1) = -4

D(1,1) = -4

Saddle Points: (-1,-1,1),(1,-1,1)

Maximize and minimize f subject to the constraint $2x^2 + 3y^2 = 1$.

Let $g(x) = 2x^2 + 3y^2 - 1$

$\nabla f(x,y) = \langle 2xy + 2x, x^2 - 1 \rangle$

$\nabla g(x,y) = \langle 4x, 6y \rangle$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$2x(y+1) = 4\lambda x$

So $x = 0$ or $y + 1 = 2\lambda$.

If $x = 0$, then $y = \pm\frac{\sqrt{3}}{3}$.

$f\left(0,\frac{\sqrt{3}}{3}\right) = -\frac{\sqrt{3}}{3}$

$f\left(0,-\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$

If $x \neq 0$, then $y + 1 = 2\lambda$.

$\lambda = \frac{y+1}{2}$

$x^2 - 1 = 6\lambda y$

$\lambda = \frac{x^2-1}{6y}$

$\frac{y+1}{2} = \frac{x^2-1}{6y}$

$6y(y + 1) = 2(x^2 - 1)$

$6y^2 + 6y = 2x^2 - 2$

$6y^2 + 6y = 1 - 3y^2 - 2$

$9y^2 + 6y + 1 = 0$

$(3y + 1)^2 = 0$

$y = \text{-}\frac{1}{3}$

$x = \pm\frac{\sqrt{3}}{3}$

$f\left(\text{-}\frac{\sqrt{3}}{3},\text{-}\frac{1}{3}\right) = \frac{5}{9}$

$f\left(\frac{\sqrt{3}}{3},\text{-}\frac{1}{3}\right) = \frac{5}{9}$

$f\left(0,\frac{\sqrt{3}}{3}\right) = \text{-}\frac{\sqrt{3}}{3}$ (min)

$f\left(0,\text{-}\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}$ (max)

(10) 2. Maximize and minimize $f(x,y) = e^{xy}$ subject to the constraint $x^2 + y^2 = 2$.

Let $g(x) = x^2 + y^2 - 2$

$\nabla f(x,y) = \langle ye^{xy}, xe^{xy}\rangle$

$\nabla g(x,y) = \langle 2x, 2y\rangle$

$\nabla f(x,y) = \lambda\nabla g(x,y)$

$ye^{xy} = 2\lambda x$

$xe^{xy} = 2\lambda y$

$\lambda = \frac{xe^{xy}}{2y} = \frac{ye^{xy}}{2x}$

$x^2 = y^2$

$x^2 + y^2 = 2$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

f(-1,-1) $= f$(1,1) $= e$ (max)

f(-1,1) $= f$(1,-1) $= \frac{1}{e}$ (min)

(10) 3. Let $z = x^2 + 3y^2$. Use differentials to approximate the change in z when (x,y) changes from (2,1) to (2.01,.99).

$dz = 2xdx + 6ydy$

$dz = 4(.01) + 6(\text{-}.01) = \text{-}.02$

(10) 4. Let $z = x^3y^2$ where $x = \sin t$ and $y = e^t$. Find $\frac{dz}{dt}$.

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 3x^2y^2(\cos t) + 2x^3ye^t = 3e^{2t}\sin^2 t \cdot \cos t + 2e^{2t}\sin^3 t$

(10) 5. For the function below, find the directional derivative of the function at the indicated point in the direction of the given vector. Recall: A unit vector is required.

$g(x,y) = e^x \sin y$, $\left(0,\frac{\pi}{6}\right)$, $\boldsymbol{v} = \langle 3,4\rangle$

Let $\boldsymbol{u} = \langle \frac{3}{5},\frac{4}{5}\rangle$

$\nabla g(x,y) = \langle e^x \sin y, e^x \cos y\rangle$

$\nabla g\left(0,\frac{\pi}{6}\right) = \left\langle \frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle$

$\left\langle \frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle \cdot \langle \frac{3}{5},\frac{4}{5}\rangle$

$\frac{3}{10} + \frac{4\sqrt{3}}{10} = \frac{3+4\sqrt{3}}{10}$

6. Evaluate each of the following integrals.

(10) a. $\iint\limits_R (xe^y)dA$, where $R = [0,1] \times [0,2]$

$$\iint\limits_R (xe^y)\,dA = \int_0^1 \left(\int_0^2 (xe^y)\,dy\right) dx = \int_0^1 \left((xe^y)\Big|_0^2 \right) dx = \int_0^1 (e^2-1)x\,dx = \tfrac{1}{2}(e^2-1)x^2\Big|_0^1$$
$$= \tfrac{1}{2}(e^2-1)$$

(10) b. $\iint\limits_R (x^2+y)\,dA$, where R is the region bounded by the triangle with vertices (0,0), (1,0), and (1,1)

$$\iint\limits_R (x^2+y)\,dA = \int_0^1 \left(\int_0^x (x^2+y)\,dy\right) dx = \int_0^1 \left(\left(x^2y + \tfrac{1}{2}y^2\right)\Big|_0^x \right) dx = \int_0^1 \left(x^3 + \tfrac{1}{2}x^2\right) dx =$$
$$\left(\tfrac{1}{4}x^4 + \tfrac{1}{6}x^3\right)\Big|_0^1 = \tfrac{1}{4} + \tfrac{1}{6} = \tfrac{5}{12}$$

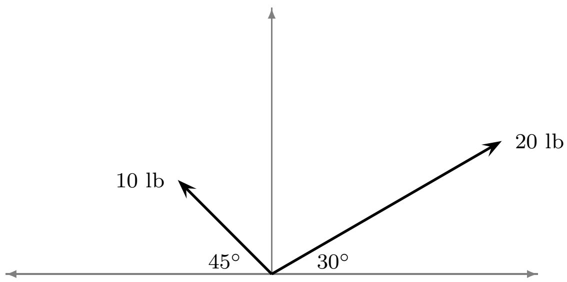
(10) c. $\iint\limits_R (x^2y)\,dA$, where R is the region bounded by the unit circle

$$\iint\limits_R (x^2y)\,dA = \int_0^{2\pi} \left(\int_0^1 r\,(r\cos\theta)^2\, r\sin\theta\,dr\right) d\theta = \int_0^{2\pi}\int_0^1 r^4\cos^2\theta\sin\theta\,dr\,d\theta = \int_0^{2\pi} \tfrac{1}{5}r^5\cos^2\theta\sin\theta\Big|_0^1\, d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{5}\cos^2\theta\sin\theta\,d\theta = \text{-}\tfrac{1}{15}\cos^3\theta\Big|_0^{2\pi}\ \text{-}\tfrac{1}{15} - \text{-}\tfrac{1}{15} = 0$$

Section 3.5: Final

Final Exam Math 2673 Spring 2017

(10) 1. Find the resultant force of the two vectors pictured below.



(10) 2. A right triangle has vertices (1,2,-3) and (2,-4,1). Give a possible third vertex. Note: There is more than one correct answer.

(10) 3. Give the equation of the plane containing the points (2,1,2), (3,4,8), (1,-2,1).

4. Let $\boldsymbol{r}(t) = \langle t^2, 2t, \ln t \rangle$.

(10) a. Give an equation of the tangent line (in any form) to the curve at the point (1,2,0).

(10) b. Give the arc length of the curve from $t = 1$ to $t = 2$.

(10) c. Find the curvature at the point (1,2,0). Hint: Use the cross product.

(10) 5. For each of the following, find the limit or prove that it does not exist.

(10) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy^2+1}$

(10) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y+x^2}{xy^2-x}$

(10) 6. Find the directional derivative of the given function at the indicated point in the direction of the given vector.

$g(x,y,z) = e^x y + \cos z,\ \left(1,2,\frac{\pi}{2}\right),\ \boldsymbol{u} = \left\langle \frac{2}{7},\frac{3}{7},\frac{6}{7} \right\rangle$

(10) 7. Find all local extrema and saddle points of the function $f(x,y) = x^2y^2 - x^2 + y^2$.

(10) 8. Maximize and minimize $f(x,y) = e^{xy}$ subject to the constraint $x^2 + y^2 = 1$.

(10) 9. Let T be the triangle in $\mathbb{R}^2$ with vertices (0,0), (4,1), (0,3) and evaluate $\displaystyle\iint_T 1\,dA$.

(10) 10. Let $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, 0 \leq x, 0 \leq y\}$ and evaluate $\displaystyle\iint_R \sqrt{x^2 + y^2}\,dA$.

11. Let S be the sphere with center (0,0,0) and radius 1 (the unit sphere) and $f(x,y,z) = \sqrt[4]{x^2 + y^2 + z^2}$.

(10) a. Express (but do not evaluate) the integral $\displaystyle\iiint_S f(x,y,z)\,dV$ as an iterated integral using rectangular coordinates.

(10) b. Express (but do not evaluate) the integral $\displaystyle\iiint_S f(x,y,z)\,dV$ as an iterated integral using cylindrical coordinates.

(10) c. Express (but do not evaluate) the integral $\iiint\limits_S f(x,y,z)\,dV$ as an iterated integral using spherical coordinates.

(10) d. Evaluate $\iiint\limits_S f(x,y,z)\,dV$ using one of the iterated integrals above.

Chapter 4: Fall 2017

Section 4.1: Quiz

Quiz 32 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the equation of the sphere with center (2,5,-1) and the following tangent plane.

a. $y = 0$

This is the xz-plane.

The distance between (2,5,-1) and the xz-plane is 5.

$(x-2)^2 + (y-5)^2 + (z+1)^2 = 25$

b. $x = 0$

This is the yz-plane.

The distance between (2,5,-1) and the yz-plane is 2.

$(x-2)^2 + (y-5)^2 + (z+1)^2 = 4$

(1) 2. Describe the region in space being defined.

$x^2 + y^2 \leq 4$, $0 \leq z \leq 5$

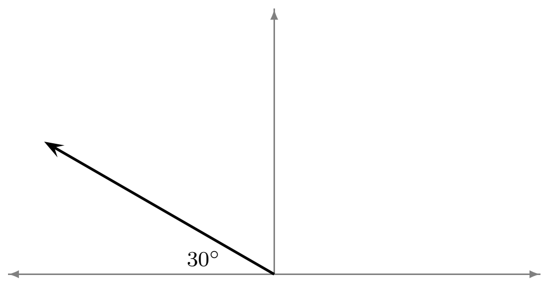
This is the solid region bounded by a right circular cylinder. The base of the cylinder is the circle in the xy-plane centered at the origin with radius 2. The height of the cylinder is 5.

Quiz 33 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. The norm of the vector pictured below is 4. Give the component form of the vector.

a.



$\boldsymbol{v} = \langle v_1, v_2 \rangle$

$\sin 30° = \frac{v_2}{\|\boldsymbol{v}\|}$

$\frac{1}{2} = \frac{v_2}{4}$

$v_2 = 2$

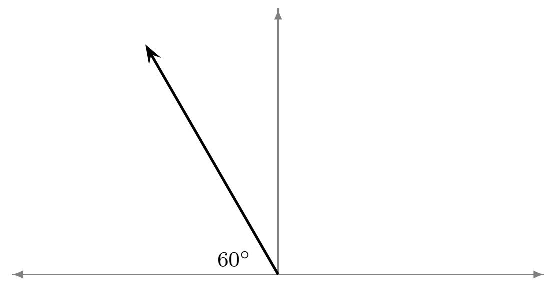
$v_1^2 + v_2^2 = 4^2$

$v_1^2 + 4 = 16$

$v_1 = \text{-}2\sqrt{3}$

$\boldsymbol{v} = \left\langle \text{-}2\sqrt{3}, 2 \right\rangle$

b.



$\boldsymbol{v} = \langle v_1, v_2 \rangle$

$\sin 60° = \frac{v_2}{\|\boldsymbol{v}\|}$

$\frac{\sqrt{3}}{2} = \frac{v_2}{4}$

$v_2 = 2\sqrt{3}$

$v_1^2 + v_2^2 = 4^2$

$v_1^2 + 12 = 16$

$v_1 = \text{-}2$

$\boldsymbol{v} = \left\langle \text{-}2, 2\sqrt{3} \right\rangle$

(1) 2. Give an example of three numbers v_1, v_2, and v_3 such that $v_1 = v_2 = v_3$ and $\| \langle v_1, v_2, v_3 \rangle \| = 1$.

Note that $\|\langle 1,1,1 \rangle\| = \sqrt{3}$. So let $v_1 = v_2 = v_3 = \frac{1}{\sqrt{3}}$.

Quiz 34 Name:________________________

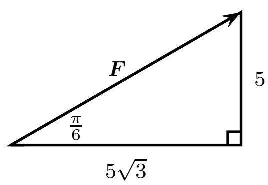
Directions: Show all of your work and justify all of your answers.

(1) 1. Find a vector orthogonal to both $\langle 1,1,0 \rangle$ and $\langle 0,1,1 \rangle$.

$\langle 1,-1,1 \rangle$

To verify, calculate the dot products.

(2) 2. A wagon is pulled 50 ft by exerting a force of 10 lb on the handle at an angle of $\frac{\pi}{6}$ with the horizontal. How much work is done?



$\boldsymbol{F} = 5\sqrt{3}\boldsymbol{i} + 5\boldsymbol{j}$

$W = \boldsymbol{F} \cdot 50\boldsymbol{i}$ ft-lb

$W = (5\sqrt{3}\boldsymbol{i} + 5\boldsymbol{j}) \cdot 100\boldsymbol{i}$ ft-lb

$W = 500\sqrt{3}$ ft-lb

Quiz 35 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{u} = \langle 1,\text{-}1,0 \rangle$, $\boldsymbol{v} = \langle 1,0,\text{-}3 \rangle$, and $\boldsymbol{w} = \langle \text{-}2,1,3 \rangle$.

(1) a. Calculate $\boldsymbol{v} \times \boldsymbol{w}$.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -3 \\ -2 & 1 & 3 \end{vmatrix} = \langle 3,3,1 \rangle$$

(1) b. Find the volume of the parallelepiped determined by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

$|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})| = \langle 1,\text{-}1,0 \rangle \cdot \langle 3,3,1 \rangle = 0$

The vectors are coplanar.

Quiz 36 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\mathcal{C}$ be the curve determined by the vector function $\boldsymbol{r}(t) = \left\langle \ln t, \frac{1}{2}t^2, \sqrt{2}t \right\rangle$.

(2) a. Give the equations of the normal plane and osculating plane of $\mathcal{C}$ at the point $\left(\ln \sqrt{2}, 1, 2\right)$.

$\boldsymbol{r}'(t) = \left\langle \frac{1}{t}, t, \sqrt{2} \right\rangle$

$\|\boldsymbol{r}'(t)\| = \sqrt{\frac{1}{t^2} + t^2 + 2} = \sqrt{\frac{t^4+2t^2+1}{t^2}} = \sqrt{\frac{(t^2+1)^2}{t^2}} = \frac{t^2+1}{t} = t + \frac{1}{t}$

$\boldsymbol{T}(t) = \frac{t}{t^2+1}\left\langle \frac{1}{t}, t, \sqrt{2} \right\rangle$

$\boldsymbol{T}'(t) = \frac{1(t^2+1)-t(2t)}{(t^2+1)^2}\left\langle \frac{1}{t}, t, \sqrt{2} \right\rangle + \frac{t}{t^2+1}\left\langle -\frac{1}{t^2}, 1, 0 \right\rangle = \frac{-t^2+1}{(t^2+1)^2}\left\langle \frac{1}{t}, t, \sqrt{2} \right\rangle + \frac{t}{t^2+1}\left\langle -\frac{1}{t^2}, 1, 0 \right\rangle$

$\boldsymbol{r}'\left(\sqrt{2}\right) = \left\langle \frac{1}{\sqrt{2}}, \sqrt{2}, \sqrt{2} \right\rangle$

$\left\|\boldsymbol{r}'\left(\sqrt{2}\right)\right\| = \frac{3}{\sqrt{2}}$

$\boldsymbol{T}\left(\sqrt{2}\right) = \frac{\sqrt{2}}{3}\left\langle \frac{1}{\sqrt{2}}, \sqrt{2}, \sqrt{2} \right\rangle = \frac{1}{3}\langle 1, 2, 2 \rangle$

$\boldsymbol{T}'\left(\sqrt{2}\right) = -\frac{1}{9}\left\langle \frac{1}{\sqrt{2}}, \sqrt{2}, \sqrt{2} \right\rangle + \frac{\sqrt{2}}{3}\left\langle -\frac{1}{2}, 1, 0 \right\rangle = -\frac{\sqrt{2}}{9}\langle 2, -2, 1 \rangle$

$\left\|\boldsymbol{T}'\left(\sqrt{2}\right)\right\| = \frac{\sqrt{2}}{3}$

$\boldsymbol{N}\left(\sqrt{2}\right) = \left(\frac{3}{\sqrt{2}} \cdot -\frac{\sqrt{2}}{9}\right)\langle 2, -2, 1 \rangle = -\frac{1}{3}\langle 2, -2, 1 \rangle$

$\boldsymbol{B}\left(\sqrt{2}\right) = \boldsymbol{T}\left(\sqrt{2}\right) \times \boldsymbol{N}\left(\sqrt{2}\right) = -\frac{1}{3}\langle 2, 1, -2 \rangle$

Normal Plane:

$\left(x - \ln \sqrt{2}\right) + 2(y-1) + 2(z-2) = 0$

Osculating Plane:

$2\left(x - \ln \sqrt{2}\right) + (y-1) - 2(z-2) = 0$

(1) b. Find the curvature of $\mathcal{C}$ at the point $\left(\ln \sqrt{2}, 1, 2\right)$.

$\kappa\left(\sqrt{2}\right) = \frac{\left\|\boldsymbol{T}'\left(\sqrt{2}\right)\right\|}{\left\|\boldsymbol{r}'\left(\sqrt{2}\right)\right\|} = \frac{\frac{\sqrt{2}}{3}}{\frac{3}{\sqrt{2}}} = \frac{2}{9}$

(1) c. Find the length of the arc from the point $\left(0, \frac{1}{2}, \sqrt{2}\right)$ to the point $\left(\ln 2\sqrt{2}, 4, 4\right)$ on $\mathcal{C}$.

$$\int_1^{2\sqrt{2}} \|\boldsymbol{r}'(t)\| \, dt = \int_1^{2\sqrt{2}} \left(t + \tfrac{1}{t}\right) dt = \left(\tfrac{1}{2}t^2 + \ln|t|\right)\Big|_1^{2\sqrt{2}} = 4 + \ln 2\sqrt{2} - \left(\tfrac{1}{2} + 0\right) = \tfrac{7}{2} + \ln 2\sqrt{2}$$

Quiz 37 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(Page 895: 21) **(2) 1.** A ball is thrown eastward into the air from the origin (in the direction of the positive x-axis). The initial velocity is $50\boldsymbol{i} + 80\boldsymbol{k}$, with speed measured in feet per second. The spin of the ball results in a southward acceleration of 4 ft/sec^2, so the acceleration vector is $\boldsymbol{a} = -4\boldsymbol{j} - 32\boldsymbol{k}$. Where does the ball land?

Quiz 38 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find the linearization of $f(x,y) = 2x^3e^y$ at the point (2,0,16). Use it to approximate $f(1.99,.01)$.

$f_x(x,y) = 6x^2e^y$

$f_y(x,y) = 2x^3e^y$

$f_x(2,0) = 24$

$f_y(2,0) = 16$

Linearization: $L(x,y) = 24(x-2) + 16y + 16$

$f(1.99,.01) \approx L(1.99,.01) = 24(1.99-2) + 16(.01) + 16 = -.24 + .16 + 16 = 15.92$

Quiz 39 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $f(x,y) = e^x \cos y$. Calculate $f'_{(0,\frac{\pi}{6})}(x,y)$.

$f_x(x,y) = e^x \cos y$

$f_y(x,y) = -e^x \sin y$

$f_x\left(0,\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f_y\left(0,\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'_{(0,\frac{\pi}{6})}(x,y) = \frac{\sqrt{3}}{2}x - \frac{1}{2}y$

Let $f(x,y) = e^y \sin x$. Calculate $f'_{(\frac{\pi}{6},0)}(x,y)$.

$f_x(x,y) = e^y \cos x$

$f_y(x,y) = e^y \sin x$

$f_x\left(\frac{\pi}{6},0\right) = \frac{\sqrt{3}}{2}$

$f_y\left(\frac{\pi}{6},0\right) = \frac{1}{2}$

$f'_{(\frac{\pi}{6},0)}(x,y) = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$

Quiz 40 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $z = xy + y$ where $x = t^2 - 3$ and $y = 2t - 6$.

(1) a. Use the chain rule to find $\frac{dz}{dt}$.

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = 2ty + 2(x+1) = 2t(2t-6) + 2(t^2 - 3 + 1) = 6t^2 - 12t - 4$

(1) b. Express z as function of t and then find $\frac{dz}{dt}$.

$z = xy + y = (t^2 - 3)(2t - 6) + (2t - 6) = 2t^3 - 6t^2 - 4t + 12$

$\frac{dz}{dt} = 6t^2 - 12t - 4$

Quiz 41 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $f(x,y,z) = x\sin y + e^{x+z}$. Find the directional derivative of f at (1,0,-1) in the direction of $\left\langle \frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$.

$\nabla f(x,y,z) = \langle \sin y + e^{x+z}, x\cos y, e^{x+z} \rangle$

$\nabla f(1,0,-1) = \langle 1,1,1 \rangle$

$\langle 1,1,1 \rangle \cdot \left\langle \frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$

$= \frac{\sqrt{2}}{2} + \frac{1}{2} + \frac{1}{2} = \frac{\sqrt{2}+2}{2}$

Quiz 42 Name:________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers.

(2) 1. Let $R = [0,1] \times [0,1]$ and calculate $\displaystyle\iint\limits_R \left(2xye^{x^2}\right) dA$.

$$\int_0^1 \int_0^1 2xye^{x^2}\,dx\,dy$$

$$= \int_0^1 ye^{x^2}\Big|_0^1\,dy$$

$$= \int_0^1 y(e-1)\,dy$$

$$= \tfrac{1}{2}y^2(e-1)\Big|_0^1$$

$$= \tfrac{1}{2}(e-1)$$

(2) 2. Let T be the triangle with vertices $(0,0)$, $\left(\sqrt{\tfrac{\pi}{2}},0\right)$, and $\left(\sqrt{\tfrac{\pi}{2}},\sqrt{\tfrac{\pi}{2}}\right)$ and calculate $\displaystyle\iint\limits_T \left(\cos x^2\right) dA$.

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^x \cos x^2\,dy\,dx$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} y\cos x^2\Big|_0^x\,dx$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} x\cos x^2\,dx$$

$$= \tfrac{1}{2}\sin x^2\Big|_0^{\sqrt{\frac{\pi}{2}}}$$

$$= \tfrac{1}{2}$$

Quiz 43 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let R be the region in the xy-plane that is inside the circle centered at (0,0) with radius 2 and outside the circle centered at (0,0) with radius 1 and calculate $\displaystyle\iint_R \left(\sqrt{x^2+y^2+1}\right) dA$.

$$\int_0^{2\pi}\int_1^2 r\sqrt{r^2+1}\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{3}\left(r^2+1\right)^{\frac{3}{2}}\Big|_1^2\, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{3}\left(5\sqrt{5}-2\sqrt{2}\right)\, d\theta$$

$$= \tfrac{1}{3}\left(5\sqrt{5}-2\sqrt{2}\right)\theta\Big|_0^{2\pi}$$

$$= \tfrac{2\pi}{3}\left(5\sqrt{5}-2\sqrt{2}\right)$$

(2) 2. Evaluate the following integral. Hint: Use polar coordinates.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} \left(x^2+y^2\right)^{\frac{3}{2}}\, dx\, dy$$

Note that $\displaystyle\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} \left(x^2+y^2\right)^{\frac{3}{2}}\, dx\, dy = \iint_R \left(x^2+y^2\right)^{\frac{3}{2}}\, dA$ where R is the right half of the circle centered at (0,0) with radius 2.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} \left(x^2+y^2\right)^{\frac{3}{2}}\, dx\, dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r\left(r^2\right)^{\frac{3}{2}}\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r^4\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{5}r^5\Big|_0^2\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{32}{5}\, d\theta$$

$$= \tfrac{32}{5}\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \tfrac{32\pi}{5}$$

Section 4.2: Exam 1

Exam 1 Math 2673 Fall 2017

Name:________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

3. Consider the points $P(2,0,1)$, $Q(5,1,0)$, and $R(1,2,2)$.

(10) a. Give the equation(s) of the line (in any form) that passes through the points P and Q.

$\boldsymbol{v} = \langle 3,1,\text{-}1\rangle$

$\boldsymbol{r}(t) = \langle 2,0,1\rangle + t\langle 3,1,\text{-}1\rangle$

(10) b. Give the equation of the plane that contains the points P, Q, and R.

$\boldsymbol{w} = \langle \text{-}4,1,2\rangle$

$\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle 3,\text{-}2,7\rangle$

(10) 4. Give the equation for the set of all points that are equidistant from the points (1,0,-1) and (2,4,1).

$(x-1)^2 + y^2 + (z+1)^2 = (x-2)^2 + (y-4)^2 + (z-1)^2$

$x^2 - 2x + 1 + y^2 + z^2 + 2z + 1 = x^2 - 4x + 4 + y^2 - 8y + 16 + z^2 - 2z + 1$

$2x + 8y + 4z = 19$

(10) 5. Are the following two vectors parallel, perpendicular, or neither?

$\boldsymbol{v} = \langle 1,\text{-}2,3\rangle$

$\boldsymbol{w} = \langle 2,\text{-}4,\text{-}5\rangle$.

No. If $\boldsymbol{w} = \alpha\boldsymbol{v}$, then $2 = \alpha \cdot 1$ which implies that $\alpha = 2$ and $\text{-}5 = \alpha \cdot 3$ which implies that $\alpha = \text{-}\frac{5}{3}$. Therefore, no such α exists.

(10) 6. Give the direction cosinses of the following vector.

$\langle \text{-}3, 0, 4\rangle$

$\|\langle \text{-}3, 0, 4\rangle\| = 5$

$\cos\alpha = \text{-}\frac{3}{5}$.

$\cos\beta = 0$

$\cos\gamma = \frac{4}{5}$.

(10) 7. Find the work done by a force of $\langle 2,1,\text{-}6\rangle$ that moves an object from (0,0,0) to (4,5,-2) along a straight line. Distance is measured in feet and force in pounds.

$\boldsymbol{D} = \langle 4,5,\text{-}2\rangle$

$\boldsymbol{F} = \langle 2,1,\text{-}6\rangle$

$\boldsymbol{D} \cdot \boldsymbol{F} = 25$

(10) 8. Determine whether the given pairs of lines are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$x = 1$

$y = 1 + 2t$

$z = 1 - 3t$

$x = 2 + s$

$y = \text{-}1 - 4s$

$z = 3 + 5s$

$2 + s = x$

$2 + s = 1$

$s = \text{-}1$

$x = 1$

$y = 3$

$z = \text{-}2$

Check that (1,3,-2) lies on both lines.

9. Describe or sketch the given surface.

(10) a. $\text{-}x^2 + y^2 + z^2 = 1$

This is a hyperboloid of one sheet with x-axis as its central axis.

(10) b. $x^2 + y^2 = 1$

This is a circular cylinder with radius 1 and having the z-axis as the central axis.

(10) c. $y = x^2 + 2z^2$

This is an elliptic paraboloid having the y-axis as the central axis.

10. Let $\boldsymbol{r}(t) = \langle e^t, \cos t, \sin t\rangle$.

(10) a. Sketch the graph of $\boldsymbol{r}$.

(10) b. Calculate the derivative of $\boldsymbol{r}$.

$\boldsymbol{r}'(t) = \langle e^t, \text{-}\sin t, \cos t\rangle$.

Section 4.3: Exam 2

Exam 2 Math 2673 Fall 2017

Name:________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

1. Let $\mathcal{C}$ be the curve defined by $\boldsymbol{r}(t) = \langle \cos t, \sin t, 2\sqrt{2}t \rangle$.

(10) a. Find the unit tangent vector $\boldsymbol{T}(t)$.

$\boldsymbol{r}'(t) = \langle \text{-}\sin t, \cos t, 2\sqrt{2} \rangle$

$\|\boldsymbol{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 8} = 3$

$\boldsymbol{T}(t) = \frac{1}{3}\langle \text{-}\sin t, \cos t, 2\sqrt{2} \rangle$

(10) b. Find the unit normal vector $\boldsymbol{N}(t)$.

$\boldsymbol{T}'(t) = \frac{1}{3}\langle \text{-}\cos t, \text{-}\sin t, 0 \rangle$

$\|\boldsymbol{T}'(t)\| = \frac{1}{3}$

$\boldsymbol{N}(t) = 3 \cdot \frac{1}{3}\langle \text{-}\cos t, \text{-}\sin t, 0 \rangle = \langle \text{-}\cos t, \text{-}\sin t, 0 \rangle$

(10) c. Find the binormal vector $\boldsymbol{B}(t)$.

$\boldsymbol{B}(t)$

$= \boldsymbol{T}(t) \times \boldsymbol{N}(t)$

$= \frac{1}{3}\langle \text{-}\cos t, \sin t, 2\sqrt{2} \rangle \times \langle \text{-}\cos t, \text{-}\sin t, 0 \rangle$

$= \frac{1}{3}\langle 2\sqrt{2}\sin t, \text{-}2\sqrt{2}\cos t, 1 \rangle$

(10) d. Find the curvature $\kappa(t)$.

$\kappa(t) = \frac{\|\boldsymbol{T}'(t)\|}{|\boldsymbol{r}'(t)\|} = \frac{\frac{1}{3}}{3} = \frac{1}{9}$

(10) e. Reparameterize the curve with respect to arc length beginning at the point $t = 0$.

$$s = \int_0^t 3\,du = 3u\Big|_0^t = 3t$$

$t = \frac{1}{3}s$

$$\boldsymbol{r}\left(\tfrac{1}{3}s\right) = \left\langle \cos\left(\tfrac{1}{3}s\right), \sin\left(\tfrac{1}{3}s\right), 2\sqrt{2}\left(\tfrac{1}{3}s\right) \right\rangle = \left\langle \cos\left(\tfrac{1}{3}s\right), \sin\left(\tfrac{1}{3}s\right), \tfrac{2\sqrt{2}}{3}s \right\rangle$$

(10) 2. A ball is thrown into the air from the origin so that its initial velocity is $25\boldsymbol{i} + 35\boldsymbol{k}$. Length is measured in feet and time is measured in seconds. The acceleration vector is $\boldsymbol{a} = 5\boldsymbol{j} - 32\boldsymbol{k}$. Find the vector function that describes motion.

$\boldsymbol{a}(t) = \langle 0, 5, \text{-}32 \rangle$

$\boldsymbol{v}(t) = \langle 0, 5t, \text{-}32t \rangle + \langle 25, 0, 35 \rangle = \langle 25, 5t, \text{-}32t + 35 \rangle$

$\boldsymbol{s}(t) = \left\langle 25t, \frac{5}{2}t^2, \text{-}16t^2 + 35t \right\rangle$

(10) 3. Give the domain and range of the following function.

$$f(x,y,z) = \frac{x^2 - y^2}{\sqrt{z}}$$

Domain: $\{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$

Range: $\mathbb{R}$

For $a \geq 0$, $f(\sqrt{a},0,1) = a$.

For $a < 0$, $f(0,\sqrt{-a},1) = a$.

(10) 4. Sketch or describe the graph of the following function.

$$f(x,y) = \sqrt{x^2 + y^2}$$

$$z = \sqrt{x^2 + y^2}$$

$$z^2 = x^2 + y^2$$

Top half of the cone $z^2 = x^2 + y^2$.

5. Calculate the following limits.

(10) a. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{y}{x^2 - y^2} = -1$

(10) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2 - y^2}$

$$\lim_{(x,0)\to(0,0)} \frac{y}{x^2 - y^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2 - 0} = 0$$

$$\lim_{(0,y)\to(0,0)} \frac{y}{x^2 - y^2} = \lim_{(0,y)\to(0,0)} \frac{y}{-y^2} = \lim_{(0,y)\to(0,0)} -\frac{1}{y}$$

$\displaystyle\lim_{(0,y)\to(0,0)} -\frac{1}{y}$ does not exist

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2 - y^2}$ does not exist

6. Let $f(x,y) = x^2y + x^2 - y$.

(10) a. Find all local extrema and saddle points.

$f_x(x,y) = 2xy + 2x = 2x(y+1)$

$f_y(x,y) = x^2 - 1 = (x+1)(x-1)$

Critical points: (-1,-1), (1,-1)

$f_{xx}(x,y) = 2y + 2$

$f_{xy}(x,y) = 2x$

$f_{yy}(x,y) = 0$

$D(x,y) = -4x^2$

$D(-1,-1) = -4$

$D(1,-1) = -4$

Saddle Points: (-1,-1,1), (1,-1,1)

(10) b. Find the absolute extrema of f on the region in $\mathbb{R}^2$ bounded by the unit circle.

The critical points (-1,-1) and (1,-1) are not in the region.

Boundary points:

$x^2 + y^2 = 1$

$x^2 = 1 - y^2$

$x = \pm\sqrt{1-y^2}$

$f\left(\sqrt{1-y^2},y\right) = f\left(-\sqrt{1-y^2},y\right) = (1-y^2)y + 1 - y^2 - y = -y^3 - y^2 + 1$

Consider $g(y) = -y^3 - y^2 + 1$ on [-1,1].

$g(-1) = f(0,-1) = 1$

$g(1) = f(0,1) = -1$

$g'(y) = -3y^2 - 2y = -y(3y+2)$

$g(0) = f(1,0) = f(-1,0) = 1$

$g\left(-\frac{2}{3}\right) = f\left(\frac{\sqrt{5}}{3},-\frac{2}{3}\right) = f\left(-\frac{\sqrt{5}}{3},-\frac{2}{3}\right) = \frac{23}{27}$

So f attains its maximum value of 1 at the points (0,-1), (-1,0), and (1,0) and its minimum value of -1 at the point (0,1).

Total Points: 120

Section 4.4: Exam 3

Exam 3 Math 2673 Fall 2017

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

1. Let $f(x,y,z) = x^2y^3 + xz$.

(10) a. Calculate the gradient of f at the point (1,-1,3).

$\nabla f(x,y,z) = \langle 2xy^3 + z, 3x^2y^2, x\rangle$

$\nabla f(1,-1,3) = \langle 1,3,1\rangle$

(10) b. Give the linear approximation of f at the point (1,-1,3) and use it to approximate $f(1.01,-1.01,2.99)$.

$L(x,y,z) = 1(x-1) + 3(y+1) + 1(z-3) + 2$

$L(1.01,-1.01,2.99) = .01 - .03 - .01 + 2 = 1.97$

(10) c. Give the directional derivative of f at the point (1,-1,3) in the direction of $\left\langle \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2} \right\rangle$.

$\langle 1,3,1\rangle \cdot \left\langle \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2} \right\rangle = \frac{\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{\sqrt{3}}{2} = \sqrt{2} + \frac{\sqrt{3}}{2}$

(10) d. Find an equation for the normal line to the level surface $f(x,y,z) = 2$ at the point (1,-1,3).

$\boldsymbol{r}(t) = \langle 1,-1,3\rangle + t\langle 1,3,1\rangle$

(10) e. Give the derivative of f at the point (a,b,c).

$f'_{(a,b,c)}(x,y,z) = \nabla f(a,b,c) \cdot \langle x,y,z\rangle = \langle 2ab^3 + c, 3a^2b^2, a\rangle \cdot \langle x,y,z\rangle = (2ab^3 + c)x + 3a^2b^2y + az$

(10) 2. Let $z = xe^y$ where $x = \sin t$ and $y = \sqrt{t}$. Find $\frac{dz}{dt}$.

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = e^y \cos t + xe^y \cdot \frac{1}{2}t^{-\frac{1}{2}} = e^{\sqrt{t}} \cos t + \frac{e^{\sqrt{t}} \sin t}{2\sqrt{t}}$

3. Let $z = xe^y$ where $x = s^2 + t$ and $y = st$.

(10) a. Find $\frac{\partial z}{\partial s}$.

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = e^y \cdot 2s + xe^y \cdot t = 2se^{st} + t(s^2+t)e^{st}$

(10) b. Find $\frac{\partial z}{\partial t}$.

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = e^y \cdot 1 + xe^y \cdot s = e^{st} + s(s^2+t)e^{st}$

(10) 4. Maximize and minimize $f(x,y) = x^2 + y^2$ subject to the constraint $x + y^2 = \frac{9}{2}$.

Let $g(x) = x + y^2 - \frac{9}{2}$.

$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla g(x,y) = \langle 1, 2y \rangle$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

$\langle 2x, 2y \rangle = \lambda \langle 1, 2y \rangle$

$2x = \lambda$ and $2y = 2y\lambda$

So $y = 0$ or $\lambda = 1$.

If $y = 0$, then $x = \frac{9}{2}$.

$f\left(\frac{9}{2}, 0\right) = \frac{81}{4}$

If $\lambda = 1$, then $x = \frac{1}{2}$ and $y = \pm 2$.

$f\left(\frac{1}{2}, -2\right) = \frac{17}{4}$ (min)

$f\left(\frac{1}{2}, 2\right) = \frac{17}{4}$ (min)

$f\left(\frac{9}{2}, 0\right) = \frac{81}{4}$ (max)

(10) 5. Maximize and minimize $f(x,y,z) = xy + yz$ subject to the constraints $x^2 + y^2 = 1$ and $yz = 1$.

Let $g(x,y,z) = x^2 + y^2 - 1$

and

$h(x,y,z) = xy - 1.$

$\nabla f(x,y,z) = \langle y, x + z, y \rangle$

$\nabla g(x,y,z) = \langle 2x, 2y, 0 \rangle$

$\nabla h(x,y,z) = \langle 0, z, y \rangle$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

$\langle y, x + z, y \rangle = \lambda \langle 2x, 2y, 0 \rangle + \mu \langle 0, z, y \rangle$

$y = 2x\lambda, \quad x + z = 2y\lambda + z\mu, \quad y = y\mu$

Since $y = y\mu$, $y = 0$ or $\mu = 1$.

However, $yz = 1$, so $y \neq 0$.

Therefore, $\mu = 1$.

$x + z = 2y\lambda + z\mu$

$x + z = 2y\lambda + z \cdot 1$

$x = 2y\lambda$

$\lambda = \frac{x}{2y}$

$y = 2x\lambda$

$\lambda = \frac{y}{2x}$

$\frac{x}{2y} = \frac{y}{2x}$

$x^2 = y^2$

$x^2 + y^2 = 1$

$2x^2 = 1$

$x = \pm \frac{\sqrt{2}}{2}$

$y = \pm \frac{\sqrt{2}}{2}$

$yz = 1$

$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right) = \frac{3}{2}$ (max)

$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right) = \frac{1}{2}$ (min)

$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\sqrt{2}\right) = \frac{1}{2}$ (min)

$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{2}\right) = \frac{3}{2}$ (max)

Alternatively:

Since $xy = 1$, $f(x,y,z) = xy + 1$. So let $g(x,y) = xy + 1$ and maximize and minimize g subject to the constraint $x^2 + y^2 = 1$.

Let $h(x) = x^2 + y^2 - 1.$

$\nabla g(x,y) = \langle y, x \rangle$

$\nabla h(x,y,z) = \langle 2x, 2y \rangle$

$\nabla g(x,y) = \lambda \nabla h(x,y)$

$y = 2x\lambda, \qquad x = 2y\lambda$

As above, $x = \pm \frac{\sqrt{2}}{2}$ and $y = \pm \frac{\sqrt{2}}{2}$.

$g\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{3}{2}$ (max)

$g\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{1}{2}$ (min)

$g\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$ (min)

$g\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{3}{2}$ (max)

Section 4.5: Final

Final Exam Math 2673 Fall 2017

Name:____________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

6. Let $\boldsymbol{u} = \langle 1,1,-1 \rangle$, $\boldsymbol{v} = \langle 1,0,-1 \rangle$, and $P = (3,-2,1)$.

(10) a. Find the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(10) b. Give the equation of the line through P in the direction of $\boldsymbol{u}$.

(10) c. Give the equation of the plane that contains P, $\boldsymbol{u}$, and $\boldsymbol{v}$.

7. Let $\boldsymbol{r}(t) = \left\langle t, 2\sqrt{2t}, \ln t \right\rangle$.

(10) a. Give an equation of the tangent line (in any form) to the curve at the point $\left(e^2, 2\sqrt{2}e, 2\right)$.

(10) b. Give the arc length of the curve from $t = 1$ to $t = e$.

(10) 8. A ball is thrown into the air from the origin so that its initial velocity is $20\boldsymbol{i} + 15\boldsymbol{k}$. Length is measured in feet and time is measured in seconds. The acceleration vector is $\boldsymbol{a} = 10\boldsymbol{j} - 32\boldsymbol{k}$. Find the vector function that describes motion.

9. Sketch or describe the following surfaces.

(10) a. $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

(10) b. $z^2 = x^2 + y^2$

(10) c. $-x^2 + y^2 - z^2 = 1$

10. Let $f(x,y) = x^2 + xy$.

(10) a. Show that f is differentiable at (a,b) for all $(a,b) \in \mathbb{R}^2$.

(10) b. Find $f'_{(a,b)}(x,y)$.

(10) 11. Maximize and minimize $f(x,y,z) = x^2+2yz$ subject to the constraint $2x + 2y + 2z = 1$.

12. Calculate $\iint\limits_R \left(4xy\sqrt{x^2+y^2}\right) dA$ for each of the following.

(10) a. $R = [0,1] \times [0,1]$

(10) b. R is the unit circle

(10) 13. Let R be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$ and calculate $\displaystyle\iiint_R x^2 y \, dV$.

(10) 14. Let R be the region in the first octant bounded by the unit sphere and calculate

$$\iiint\limits_R e^{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\, dV.$$

Final Exam Math 2673 Fall 2017

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

15. Let $\boldsymbol{u} = \langle 1,1,-1 \rangle$, $\boldsymbol{v} = \langle 1,0,1 \rangle$, and $P = (1,-2,3)$.

(10) a. Find the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(10) b. Give the equation of the line through P in the direction of $\boldsymbol{u}$.

(10) c. Give the equation of the plane that contains P, $\boldsymbol{u}$, and $\boldsymbol{v}$.

16. Let $\boldsymbol{r}(t) = \left\langle t, 2\sqrt{2t}, \ln t \right\rangle$.

(10) a. Give an equation of the tangent line (in any form) to the curve at the point $\left(e^2, 2\sqrt{2}e, 2\right)$.

(10) b. Give the arc length of the curve from $t = 1$ to $t = e$.

(10) c. Calculate the curvature at the point $\left(e^2, 2\sqrt{2}e, 2\right)$.

17. Sketch or describe the following surfaces.

(10) a. $\frac{y^2}{9} + \frac{z^2}{4} = 1 - x^2$

(10) b. $z^2 = x^2 - y^2$

(10) c. $x^2 - y^2 - z^2 = 1$

18. Let $f(x,y) = x + xy^2$.

(10) a. Show that f is differentiable at (a,b) for all $(a,b) \in \mathbb{R}^2$.

(10) b. Find $f'_{(a,b)}(x,y)$.

(10) 19. Maximize and minimize $f(x,y,z) = x^2 + 2yz$ subject to the constraint $2x + 2y + 2z = 1$.

20. Calculate $\iint\limits_R \left(4xy\sqrt{x^2+y^2}\right)dA$ for each of the following.

(10) a. $R = [0,1] \times [0,1]$

(10) b. R is the unit circle

(10) 21. Let R be the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$ and calculate $\iiint\limits_R x^2 y \, dV$.

(10) 22. Let R be the region in the first octant bounded by the unit sphere and calculate $\iiint\limits_R \sqrt{(x^2 + y^2 + z^2)^3}\, dV.$

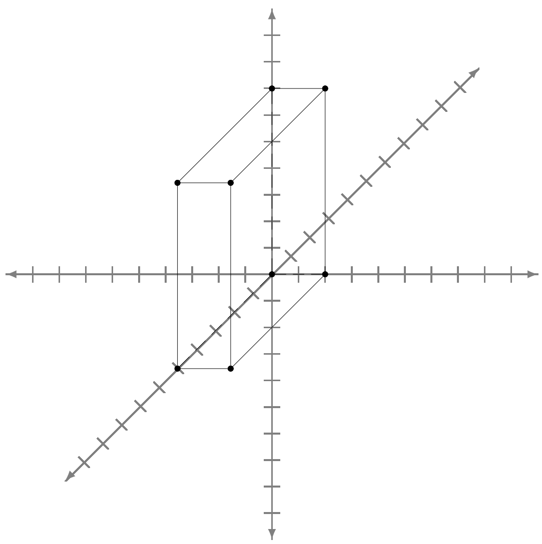
Chapter 5: Fall 2018

Section 5.1: Quizzes

Quiz 44 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. The bottom of the rectangular prism below lies in the xy-plane and its height is 7. Give the coordinates of the eight corners.



(0,0,0)

(5,0,0)

(0,2,0)

(5,2,0)

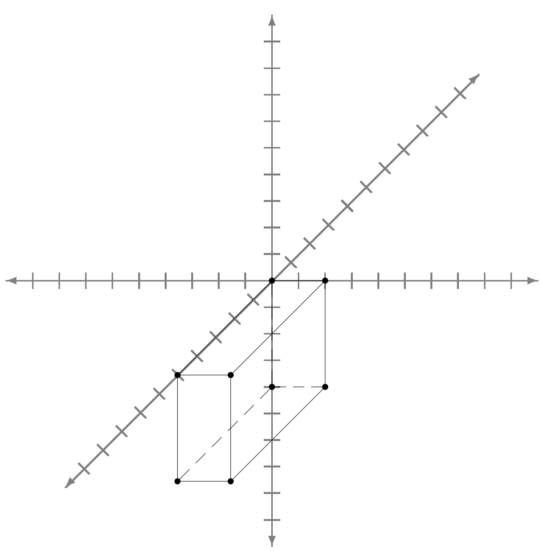
(0,0,7)

(5,0,7)

(0,2,7)

(5,2,7)

2. The top of the rectangular prism below lies in the xy-plane and its height is 4. Give the coordinates of the eight corners.



(0,0,0)

(5,0,0)

(0,2,0)

(5,2,0)

(0,0,-4)

(5,0,-4)

(0,2,-4)

(5,2,-4)

(1) 3. Give the equation of the sphere with center (-1,4,2) and radius $\sqrt{3}$.

$(x+1)^2 + (y-4)^2 + (z-2)^2 = 3$

4. Give the equation of the sphere with center (-1,4,3) and radius $\sqrt{2}$.

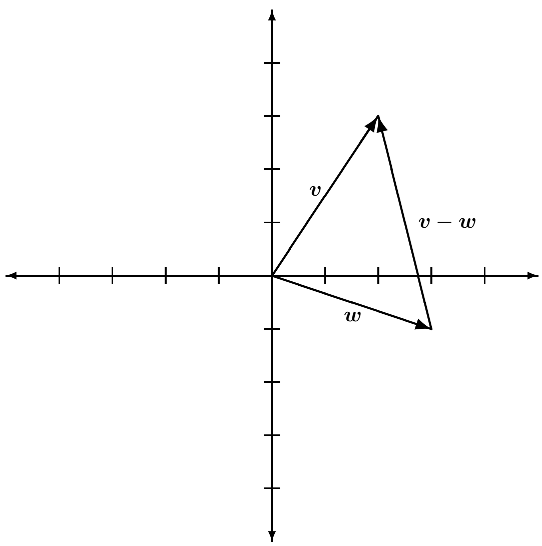
$(x+1)^2 + (y-4)^2 + (z-3)^2 = 2$

Quiz 45 Name:________________________

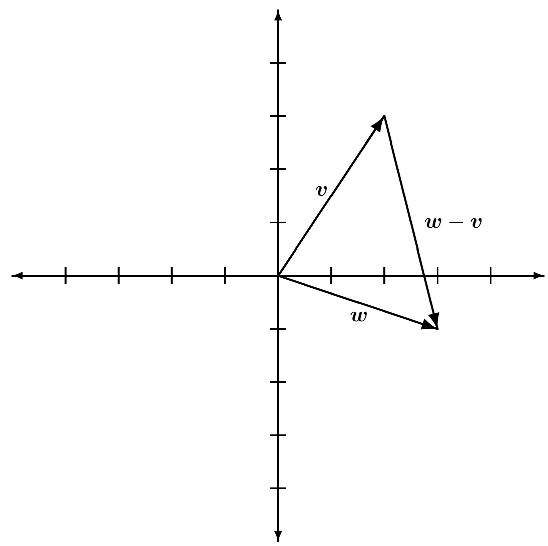
Directions: Show all of your work and justify all of your answers.

(1) 1.

In the figure below, draw the vector $\boldsymbol{v} - \boldsymbol{w}$.



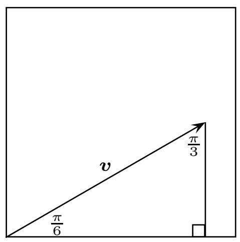
In the figure below, draw the vector $\boldsymbol{w} - \boldsymbol{v}$.



Quiz 46 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. A spider is in the southwest corner of a room that is $10\sqrt{3}$ ft by $10\sqrt{3}$ ft. The spider walks along a straight line that makes an angle of $\frac{\pi}{6}$ radians with the south wall. The spider is walking at a constant rate of 2 ft/min. Where and when will the spider reach another wall? Hint: Give the position of the spider as a vector determined by time.



$\|\boldsymbol{v}\| = 2t$

$\boldsymbol{v} = \langle \sqrt{3}t, t \rangle$

The spider reaches the east wall when $\sqrt{3}t = 10\sqrt{3}$.

So after 10 minutes the spider's position relative to the southwest corner is $\langle 10\sqrt{3}, 10 \rangle$ which is on the east wall and 10 ft from the south wall.

Quiz 47 Name:________________________

Directions: Show all of your work and justify all of your answers.

(4) 1. Let $\boldsymbol{x} = \langle 1, \text{-}1, 1\rangle$, $\boldsymbol{y} = \langle 1, 1, 2\rangle$, and $\boldsymbol{v} = \langle 3, 1, \text{-}2\rangle$. Also, let θ be the angle between $\boldsymbol{x}$ and $\boldsymbol{v}$. Find each of the following.

a. $\|\boldsymbol{x}\| = \sqrt{3}$

b. $\|\boldsymbol{v}\| = \sqrt{14}$

c. $\text{comp}_{\boldsymbol{v}}\boldsymbol{x} = \frac{0}{\sqrt{14}} = 0$

d. $\text{proj}_{\boldsymbol{v}}\boldsymbol{x} = \mathbf{0}$

e. $\boldsymbol{x} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & \text{-}1 & 1 \\ 3 & 1 & \text{-}2 \end{vmatrix} = \langle 1{,}5{,}4\rangle$

f. $\cos\theta = \frac{0}{\sqrt{3}\sqrt{14}} = 0$

g. $\sin\theta = \frac{\|\langle 1,5,4\rangle\|}{\sqrt{3}\sqrt{14}} = \frac{\sqrt{42}}{\sqrt{42}} = 1$

h. The volume of the parallelpiped determined by the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{v}$.

$$\left\| \begin{matrix} 1 & \text{-}1 & 1 \\ 1 & 1 & 2 \\ 3 & 1 & \text{-}2 \end{matrix} \right\| = |1(\text{-}2 - 2) + 1(\text{-}2 - 6) + 1(1 - 3)| = 14$$

Quiz 47 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(4) 1. Let $\boldsymbol{x} = \langle 1, \text{-}1, 1\rangle$, $\boldsymbol{y} = \langle 1, 1, 2\rangle$, and $\boldsymbol{v} = \langle 3, 1, \text{-}2\rangle$. Also, let θ be the angle between $\boldsymbol{y}$ and $\boldsymbol{v}$. Find each of the following.

a. $\|\boldsymbol{y}\| = \sqrt{6}$

b. $\|\boldsymbol{v}\| = \sqrt{14}$

c. $\text{comp}_{\boldsymbol{v}}\boldsymbol{y} = \frac{0}{\sqrt{14}} = 0$

d. $\text{proj}_{\boldsymbol{v}}\boldsymbol{y} = \mathbf{0}$

e. $\boldsymbol{y} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 2 \\ 3 & 1 & \text{-}2 \end{vmatrix} = \langle \text{-}4, 8, \text{-}2\rangle$

f. $\cos\theta \; \frac{0}{\sqrt{6}\sqrt{14}} = 0$

g. $\sin\theta = \frac{\|\langle \text{-}4,8,\text{-}2\rangle\|}{\sqrt{6}\sqrt{14}} = \frac{\sqrt{84}}{\sqrt{84}} = 1$

h. The volume of the parallelpiped determined by the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{v}$.

$$\left\| \begin{matrix} 1 & \text{-}1 & 1 \\ 1 & 1 & 2 \\ 3 & 1 & \text{-}2 \end{matrix} \right\| = |1(\text{-}2 - 2) + 1(\text{-}2 - 6) + 1(1 - 3)| = 14$$

Quiz 48 Name:________________________

Directions: Show all of your work and justify all of your answers.

1.

Let $\boldsymbol{r}(t) = \left\langle \sqrt{t}+1, \sin(\pi t), \frac{t}{e^t} \right\rangle$.

(1) a. Calculate $\lim_{t\to 0} \boldsymbol{r}(t)$.

$\lim_{t\to 0} \boldsymbol{r}(t)$

$= \lim_{t\to 0} \left\langle \sqrt{t}+1, \sin(\pi t), \frac{t}{e^t} \right\rangle$

$= \langle 1{,}0{,}0 \rangle$

(1) b. Find $\boldsymbol{r}'(t)$.

$\boldsymbol{r}'(t)$

$= \left\langle \frac{1}{2\sqrt{t}}, \pi\cos(\pi t), \frac{e^t - te^t}{e^{2t}} \right\rangle$

$= \left\langle \frac{1}{2\sqrt{t}}, \pi\cos(\pi t), \frac{1-t}{e^t} \right\rangle$

(1) c. Give the equation(s) of the line that is tangent to the graph of $\boldsymbol{r}$ at the point where $t = 4$.

$\boldsymbol{r}(4) = \left\langle 3{,}0{,}\frac{4}{e^4} \right\rangle$

$\boldsymbol{r}'(4) = \left\langle \frac{1}{4}{,}\pi{,}\frac{-3}{e^4} \right\rangle$

$\boldsymbol{l}(t) = \left\langle 3{,}0{,}\frac{4}{e^4} \right\rangle + t\left\langle \frac{1}{4}{,}\pi{,}\frac{-3}{e^4} \right\rangle$

Let $\boldsymbol{r}(t) = \left\langle \sqrt{t}+1, -\cos(\pi t), \frac{t}{e^t} \right\rangle$.

$\lim_{t\to 0} \boldsymbol{r}(t)$

$= \lim_{t\to 0} \left\langle \sqrt{t}+1, -\cos(\pi t), \frac{t}{e^t} \right\rangle$

$= \langle 1{,}-1{,}0 \rangle$

$\boldsymbol{r}'(t)$

$= \left\langle \frac{1}{2\sqrt{t}}, \pi\sin(\pi t), \frac{e^t - te^t}{e^{2t}} \right\rangle$

$= \left\langle \frac{1}{2\sqrt{t}}, \pi\sin(\pi t), \frac{1-t}{e^t} \right\rangle$

$\boldsymbol{r}(4) = \left\langle 3{,}-1{,}\frac{4}{e^4} \right\rangle$

$\boldsymbol{r}'(4) = \left\langle \frac{1}{4}{,}0{,}\frac{-3}{e^4} \right\rangle$

$\boldsymbol{l}(t) = \left\langle 3{,}-1{,}\frac{4}{e^4} \right\rangle + t\left\langle \frac{1}{4}{,}0{,}\frac{-3}{e^4} \right\rangle$

Quiz 49 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Calculate the following integral.

$$\int_0^1 \left\langle \cos(\pi t), e^t, \tfrac{1}{t^2+1} \right\rangle dt$$

$$= \left\langle \tfrac{1}{\pi}\sin(\pi t), e^t, \tan^{-1} t \right\rangle \Big|_0^1$$

$$= \left\langle 0, e, \tfrac{\pi}{4} \right\rangle - \langle 0,1,0 \rangle$$

$$= \left\langle 0, e-1, \tfrac{\pi}{4} \right\rangle$$

2. Consider the vector function $\boldsymbol{r}(t) = \left\langle e^t, \sqrt{2}t, e^{-t} \right\rangle$.

(2) a. Calculate and simplify $\|\boldsymbol{r}'(t)\|$.

$$\boldsymbol{r}'(t) = \left\langle e^t, \sqrt{2}, -e^{-t} \right\rangle$$

$$\|\boldsymbol{r}'(t)\|$$

$$= \sqrt{e^{2t} + 2 + e^{-2t}}$$

$$= \sqrt{\tfrac{e^{4t}+2e^{2t}+1}{e^{2t}}}$$

$$= \sqrt{\tfrac{(e^{2t}+1)^2}{e^{2t}}}$$

$$= \tfrac{e^{2t}+1}{e^t}$$

$$= e^t + e^{-t}$$

(1) b. Give the arc length of the curve from $t = 0$ to $t = 1$.

$$\int_0^1 \left(e^t + e^{-t}\right) dt$$

$$= \left(e^t - e^{-t}\right)\Big|_0^1$$

$$= e - \tfrac{1}{e} - (1-1)$$

$$= e - \tfrac{1}{e}$$

Quiz 50 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\mathcal{C}$ be the curve defined by $\boldsymbol{r}(t) = \langle t^2, t, 1 \rangle$.

(1) a. Find the unit tangent vector.

$\boldsymbol{r}'(t) = \langle 2t,1,0 \rangle$

$\|\boldsymbol{r}'(t)\| = \sqrt{4t^2+1}$

$\boldsymbol{T}(t) = \frac{1}{\sqrt{4t^2+1}} \langle 2t, 1, 0 \rangle$

(1) b. Find the derivative of the unit tangent vector.

$\boldsymbol{T}(t) = (4t^2+1)^{-\frac{1}{2}} \langle 2t,1,0 \rangle$

$\boldsymbol{T}'(t)$

$= -\frac{1}{2}(4t^2+1)^{-\frac{3}{2}} (8t) \langle 2t,1,0 \rangle + (4t^2+1)^{-\frac{1}{2}} \langle 2,0,0 \rangle$

$= \frac{1}{\sqrt{(4t^2+1)^3}} [(-4t) \langle 2t,1,0 \rangle + (4t^2+1) \langle 2,0,0 \rangle]$

$= \frac{2}{\sqrt{(4t^2+1)^3}} [\langle -4t^2,-2t,0 \rangle + \langle 4t^2+1,0,0 \rangle]$

$= \frac{2}{\sqrt{(4t^2+1)^3}} \langle 1,-2t,0 \rangle$

(1) c. Find the curvature at the point (4,2,1).

$\boldsymbol{r}'(t) = \langle 2t,1,0 \rangle$

$\boldsymbol{r}''(t) = \langle 2,0,0 \rangle$

$\boldsymbol{r}'(2) \times \boldsymbol{r}''(2) = \langle 4,1,0 \rangle \times \langle 2,0,0 \rangle = \langle 0,0,-2 \rangle$

$\|\boldsymbol{r}'(2) \times \boldsymbol{r}''(2)\| = 2$

$\|\boldsymbol{r}'(2)\| = \sqrt{17}$

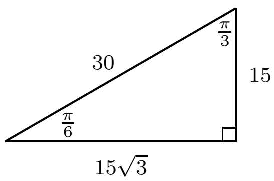
$\kappa(2) = \frac{2}{17\sqrt{17}}$

Quiz 51 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. A ball is thrown with an initial speed of 30 feet per second at an angle of $\frac{\pi}{6}$ to the horizontal. Assume that the only force acting on the object is gravity.

(1) a. Find the initial velocity vector.



$\boldsymbol{v}(0) = 15\sqrt{3}\boldsymbol{i} + 15\boldsymbol{j}$

(1) b. Find the vector functions that describe velocity and motion.

$\boldsymbol{a}(t) = \text{-}32\boldsymbol{j}$

$\boldsymbol{v}(t) = \text{-}32t\boldsymbol{j} + \boldsymbol{v}(0)$

$\boldsymbol{v}(t) = 15\sqrt{3}\boldsymbol{i} + (\text{-}32t + 15)\boldsymbol{j}$

$\boldsymbol{s}(t) = 15\sqrt{3}t\boldsymbol{i} + (\text{-}16t^2 + 15t)\boldsymbol{j}$

(1) c. Find the maximum height.

$\text{-}32t + 15 = 0$

$t = \frac{15}{32}$

$\text{-}16 \cdot \left(\frac{15}{32}\right)^2 + 15 \cdot \frac{15}{32} = \frac{225}{64}$

(1) d. Find the horizontal range.

$\text{-}16t^2 + 15t = 0$

$t = \frac{15}{16}$

$15\sqrt{3} \cdot \frac{15}{16} = \frac{225\sqrt{3}}{16}$

(1) e. Find the speed of impact.

$\left\|\boldsymbol{v}\left(\frac{15}{16}\right)\right\|$

$= \left\|15\sqrt{3}\boldsymbol{i} + (\text{-}32 \cdot \frac{15}{16} + 15\boldsymbol{j}\right\|$

$= 30$

Quiz 52 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $f(x,y) = x^2\sqrt[3]{y}$.

(1) a. Find the linearization of f at the point (10,125).

$f_x(x,y) = 2x\sqrt[3]{y}$

$f_y(x,y) = x^2 \frac{1}{3} y^{-\frac{2}{3}} = \frac{x^2}{3\sqrt[3]{y^2}}$

$f(10,125) = 500$

$f_x(10,125) = 100$

$f_y(10,125) = \frac{4}{3}$

$L(x,y) = 500 + 100(x - 10) + \frac{4}{3}(y - 125)$

(1) b. Use your answer from the previous part to estimate $(9.9)^2\sqrt[3]{126}$.

$f(9.9,126)$

$\approx L(9.9,126)$

$= 500 + 100(9.9 - 10) + \frac{4}{3}(126 - 125)$

$= 500 - 10 + \frac{4}{3}$

$= \frac{1474}{3}$

Quiz 53 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $f(x,y) = xy^2 + y$.

(1) a. Assuming that f is differentiable at (a,b) for all $(a,b) \in \mathbb{R}^2$, find $f'_{(a,b)}(x,y)$.

$f_x(x,y) = y^2$

$f_y(x,y) = 2xy + 1$

$f'_{(a,b)}(x,y) = b^2x + (2ab+1)y$

(2) b. Show that f is differentiable at (a,b) for all $(a,b) \in \mathbb{R}^2$.

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - \left[f_x(a,b)\cdot h + f_y(a,b)\cdot k\right]}{\|(h,k)\|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{(a+h)(b+k)^2 + (b+k) - (ab^2+b) - [b^2h + (2ab+1)k]}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{(a+h)(b^2+2bk+k^2) + b + k - ab^2 - b - b^2h - 2abk - k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{ab^2 + 2abk + ak^2 + b^2h + 2bhk + hk^2 + b + k - ab^2 - b - b^2h - 2abk - k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{ak^2 + 2bhk + hk^2}{\sqrt{h^2+k^2}}$$

$$\lim_{(h,k)\to(0,0)} \left|\frac{ak^2 + 2bhk + hk^2}{\sqrt{h^2+k^2}}\right| = \lim_{(h,k)\to(0,0)} \frac{|ak^2 + 2bhk + hk^2|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|k||ak + 2bh + hk|}{\sqrt{h^2+k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{|k||ak + 2bh + hk|}{\sqrt{k^2}} = \lim_{(h,k)\to(0,0)} \frac{|k||ak + 2bh + hk|}{|k|} = \lim_{(h,k)\to(0,0)} |ak + 2bh + hk| = 0$$

Quiz 54 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $z = xe^y$.

(1) a. Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$\frac{\partial z}{\partial x} = e^y$

$\frac{\partial z}{\partial y} = xe^y$

(1) b. Suppose that $x = t^2$ and $y = \sin t$. Calculate $\frac{dz}{dt}$.

$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = e^y(2t) + xe^y \cos t = 2te^{\sin t} + t^2 e^{\sin t} \cos t$

(1) c. Suppose that $x = s^2 + t$ and $y = st$. Calculate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = e^y(2s) + xe^y(t) = 2se^{st} + t(s^2 + t)e^{st}$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = e^y(1) + xe^y(s) = e^{st} + s(s^2 + t)e^{st}$

Quiz 55 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $f(x,y,z) = x^2y + xe^z$, $c = (1,\text{-}1,0)$ $[(\text{-}1,1,0)]$, $\boldsymbol{v} = \langle 2,2,1\rangle$, and $\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$.

(1) a. Calculate each of the following.

i. $\nabla f(c)$

$\nabla f(x,y,z) = \langle 2xy + e^z, x^2, xe^z\rangle$

$\nabla f(1,\text{-}1,0) = \langle \text{-}1,1,1\rangle$

$\nabla f(\text{-}1,1,0) = \langle \text{-}1,1,\text{-}1\rangle$

ii. $D_{\boldsymbol{u}}f(c)$

$\boldsymbol{u} = \frac{1}{3}\boldsymbol{v} = \frac{1}{3}\langle 2,2,1\rangle$

$\langle \text{-}1,1,1\rangle \cdot \frac{1}{3}\langle 2,2,1\rangle = \frac{1}{3}$

$\langle \text{-}1,1,\text{-}1\rangle \cdot \frac{1}{3}\langle 2,2,1\rangle = \text{-}\frac{1}{3}$

(1) b. Let $\mathcal{S}$ be the surface $f(x,y,z) = 0$. Give the equation of each of the following at the point c.

i. The tangent plane to $\mathcal{S}$.

$\text{-}1(x-1) + 1(y+1) + 1(z-0) = 0$

$\text{-}x + y + z = \text{-}2$

$\text{-}1(x+1) + 1(y-1) - 1(z-0) = 0$

$\text{-}x + y - z = 2$

ii. The normal line to $\mathcal{S}$.

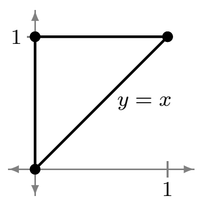
$\boldsymbol{l}(t) = \langle 1,\text{-}1,0\rangle + t\langle \text{-}1,1,1\rangle$

$\boldsymbol{l}(t) = \langle \text{-}1,1,0\rangle + t\langle \text{-}1,1,\text{-}1\rangle$

Quiz 56 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let R be the the region bounded by the triangle with vertices (0,0), (0,1), and (1,1). Express $\iint_R x \sin y^3 \, dA$ as an iterated integral for each order of integration. Evaluate one of the iterated integrals. Hint: One iterated integral is more difficult than the other.



Type I

$$\int_0^1 \int_x^1 x \sin y^3 \, dy \, dx$$

Type II

$$\int_0^1 \int_0^y x \sin y^3 \, dx \, dy$$

Type II

$$\int_0^1 \int_0^y x \sin y^3 \, dx \, dy$$

$$= \int_0^1 \frac{1}{2} x^2 \sin y^3 \Big|_0^y \, dy$$

$$= \int_0^1 \frac{1}{2} y^2 \sin y^3 \, dy$$

$$= -\frac{1}{6} \cos y^3 \Big|_0^1$$

$$= -\frac{1}{6}(\cos 1 - 1)$$

$$= \frac{1}{6}(1 - \cos 1)$$

Quiz 57 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Calculate the following integral. Hint: Sketch the region of integration and convert to polar coordinates.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} xy^2\,dy\,dx$$

$$= \int_0^{\pi}\int_0^1 r\cos\theta(r\sin\theta)^2 r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^1 r^4\cos\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi} \tfrac{1}{5}r^5\cos\theta\sin^2\theta\Big|_0^1\,d\theta$$

$$= \int_0^{\pi} \tfrac{1}{5}\cos\theta\sin^2\theta\,d\theta$$

$$= \tfrac{1}{15}\sin^3\theta\Big|_0^{\pi}$$

$$= 0$$

Quiz 58 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let R be the region bounded by the planes $x = 1$, $y = 0$, $y = x$, $z = 0$, and $z = y$. Calculate the following integral.

$$\iiint_R 72x^2 z e^{y^3}\, dV$$

$$= \int_0^1 \int_0^x \int_0^y 72x^2 z e^{y^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^x 36x^2 z^2 e^{y^3} \Big|_0^y \, dy\, dx$$

$$= \int_0^1 \int_0^x 36x^2 y^2 e^{y^3}\, dy\, dx$$

$$= \int_0^1 12x^2 e^{y^3} \Big|_0^x \, dx$$

$$= \int_0^1 \left(12x^2 e^{x^3} - 12x^2\right) dx$$

$$= \left(4e^{x^3} - 4x^3\right)\Big|_0^1$$

$$= 4e - 4 - 4$$

$$= 4(e - 2)$$

Quiz 59 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Let R be the region between the planes $z = 0$ and $z = 1$ that lies above the cone $z^2 = x^2 + y^2$. Express $\displaystyle\iiint_R xyz\,dV$ in terms of rectangular, cylindrical, and spherical coordinates. Evaluate all of the integrals. Follow the directions given in class.

Rectangular:

$$\iiint_R xyz\,dV$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} xyz\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\cdot\tfrac{1}{2}z^2\Big|_{\sqrt{x^2+y^2}}^{1}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \tfrac{1}{2}xy[1-(x^2+y^2)]\,dy\,dx$$

$$= \tfrac{1}{2}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(xy - x^3y - xy^3\right)dy\,dx$$

$$= \tfrac{1}{2}\int_{-1}^{1} \left(\tfrac{1}{2}xy^2 - \tfrac{1}{2}x^3y^2 - \tfrac{1}{4}xy^4\right)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= \tfrac{1}{2}\int_{-1}^{1} 0\,dx$$

$$= 0$$

Cylindrical:

$$\iiint_R xyz\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 r(r\cos\theta)(r\sin\theta)z\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_4^1 r^3(\cos\theta\sin\theta)z\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \tfrac{1}{2}r^3(\cos\theta\sin\theta)z^2\Big|_r^1\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}\int_0^1 (r^3 - r^5)(\cos\theta\sin\theta)\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} (\cos\theta\sin\theta)\left(\tfrac{1}{4}r^4 - \tfrac{1}{6}r^6\right)\Big|_0^1\,d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \tfrac{1}{12}(\cos\theta\sin\theta)\,d\theta$$

$$= \tfrac{1}{24}\int_0^{2\pi} \cos\theta\sin\theta\,d\theta$$

$$= \tfrac{1}{48}\sin^2\theta\Big|_0^{2\pi}$$

$$= 0$$

Spherical:

$$\iiint_R xyz\,dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\sec\phi} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\sec\phi} \rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \tfrac{1}{6}\rho^6\sin^3\phi\cos\phi\sin\theta\cos\theta\Big|_0^{\sec\phi}\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \tfrac{1}{6}\sec^6\phi\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \tfrac{1}{6}\sec^2\phi\tan^3\phi\sin\theta\cos\theta\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{12}\sec^2\phi\tan^3\phi\sin^2\theta\Big|_0^{2\pi}\,d\phi$$

$$= \int_0^{\frac{\pi}{2}} 0\,d\phi$$

$$= 0$$

Section 5.2: Exam 1

Exam 1 Math 2673 Fall 2018

(10) 2. Give the center and radius of the following sphere.

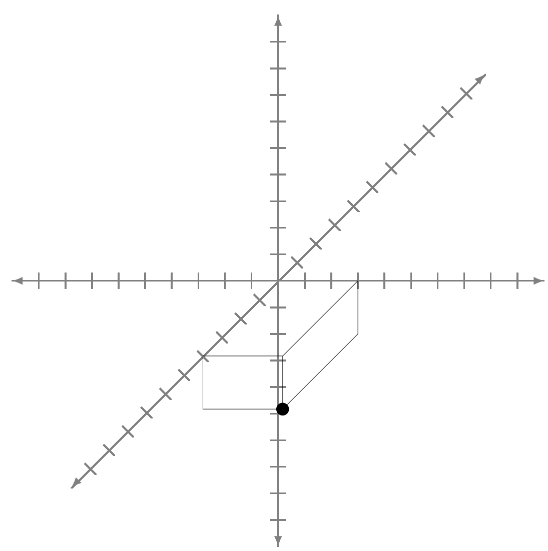
$x^2 - 2x + y^2 + z^2 + 6z + 6 = 0$

$(x-1)^2 + y^2 + (z+3)^2 = -6 + 1 + 9 = 4$

Center: $(1, 0, -3)$

Radius: 2

(10) 3. Graph the point (4,3,-2).



(10) 4. Suppose $\boldsymbol{v} \in \boldsymbol{V}_2$, $\|\boldsymbol{v}\| = 10$, and the angle between $\boldsymbol{v}$ and the positive x-axis is $\frac{\pi}{6}$ radians. Give the component form of $\boldsymbol{v}$.

$$\boldsymbol{v} = 10\left(\cos\tfrac{\pi}{6}\right)\boldsymbol{i} + 10\left(\sin\tfrac{\pi}{6}\right)\boldsymbol{j} = 5\sqrt{3}\boldsymbol{i} + 5\boldsymbol{j} = \left\langle 5\sqrt{3}, 5\right\rangle$$

5. Let $\boldsymbol{v} = \langle 1, \text{-}2, 2\rangle$, $\boldsymbol{w} = \langle 1, 4, 1\rangle$, and θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Give each of the following.

(10) a. $\boldsymbol{v} \cdot \boldsymbol{w} = 1 - 8 + 2 = \text{-}5$

(10) b. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{-5}{3\sqrt{18}} = \text{-}\frac{5}{9\sqrt{2}}$

(10) c. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \text{-}\frac{5}{3}$

(10) d. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v} = \text{-}\frac{5}{9}\boldsymbol{v} = \text{-}\frac{5}{9}\langle 1, \text{-}2, 2\rangle$

6. Consider the points $P(1,\text{-}2,0)$, $Q(\text{-}4,3,8)$, and $R(2,1,3)$.

(10) a. Give the equation of the plane containing the points P, Q, and R.

$\overrightarrow{PQ} = \langle \text{-}5,5,8\rangle$

$\overrightarrow{PR} = \langle 1,3,3\rangle$

$\boldsymbol{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle \text{-}9,23,\text{-}20\rangle$

$\text{-}9(x-1) + 23(y+2) - 20(z-0) = 0$

$\text{-}9x + 23y - 20z = \text{-}55$

(10) b. Give the equation(s) of the line containing the points P and Q.

$\overrightarrow{PQ} = \langle$-5,5,8$\rangle$

Parametric

$x = 1 - 5t$

$y =$ -2 $+ 5t$

$z = 8t$

Symmetric

$\frac{x-1}{\text{-5}} = \frac{y+2}{5} = \frac{z}{8}$

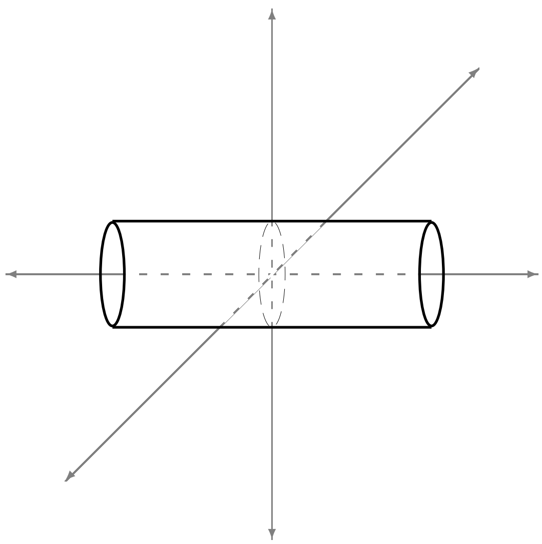
Vector

$\boldsymbol{l}(t) = \langle$1,-2,0$\rangle + t\langle$-5,5,8$\rangle$

7. For each of the following, describe and sketch the surface.

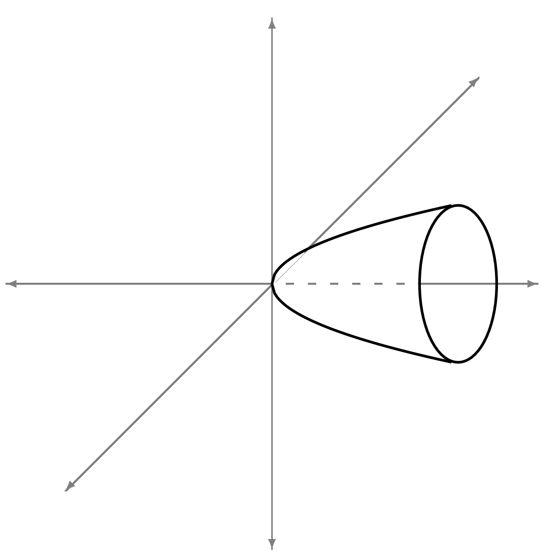
(10) a. $x^2 + z^2 = 1$

Circular cylinder whose axis is the y-axis.



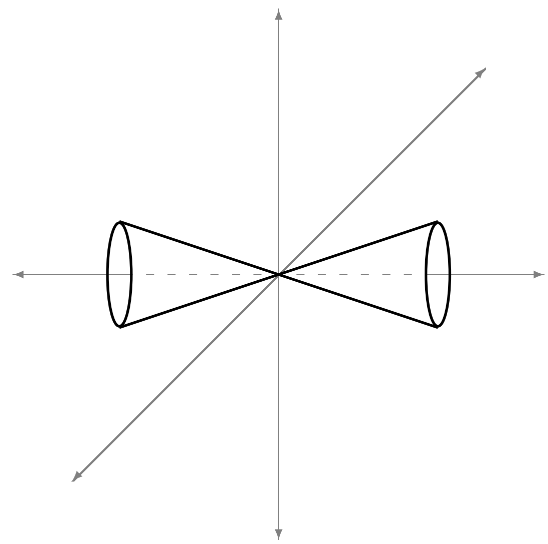
(10) b. $x^2 + z^2 = y$

Circular paraboloid whose axis is the y-axis.



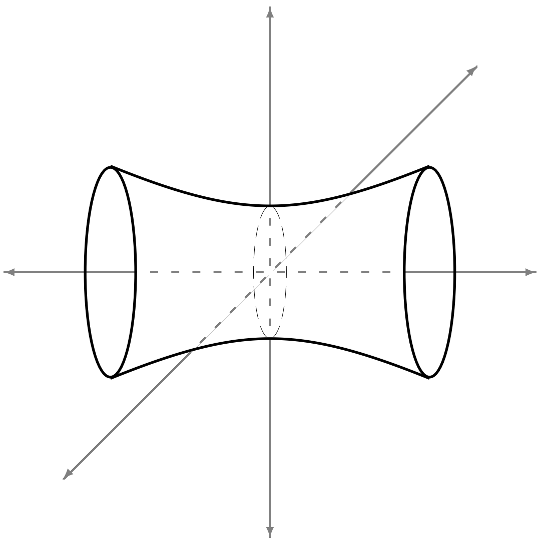
(10) c. $x^2 + z^2 = y^2$

Cone whose axis is the y-axis.



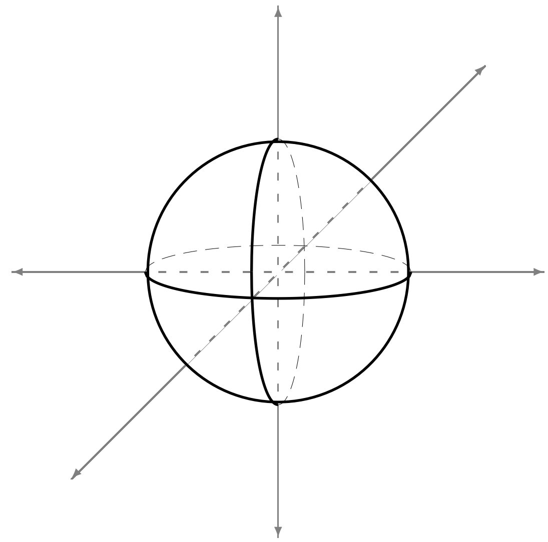
(10) d. $x^2 - y^2 + z^2 = 1$

Hyperboloid of one sheet whose axis is the y-axis.



(10) e. $x^2 + y^2 + z^2 = 1$

The unit sphere.



(10) 8. Two chipmunks are sitting at the same place at the bottom of a hill. They begin moving at the same time. The trajectory of the first chipmunk is $\boldsymbol{r}_1(t) = \langle 5t, t^2, t\rangle$. The trajectory of the second chipmunk is $\boldsymbol{r}_2(t) = \langle t^2 + t, 4t, t\rangle$. Will the chipmunks meet again? If so, when and where?

Consider the second component of each function and solve the following.

$t^2 = 4t$

$t = 0$, $t = 4$

Note that $\boldsymbol{r}_1(4) = \langle 20,16,4\rangle$ and $\boldsymbol{r}_2(4) = \langle 20,16,4\rangle$.

So the chipmunks will meet again at time $t = 4$ at the point (20,16,4).

Section 5.3: Exam 2

Exam 2 Math 2673 Fall 2018

9. A projectile is launched E30°N at an angle of elevation of 45° at a speed of 50 ft/sec. The acceleration vector is ⟨5,4,-32⟩.

(10) a. Find and simplify the initial velocity vector.

$\boldsymbol{v}_0 = \langle 50\cos 120°, 50\cos 30°, 50\cos 45°,\rangle = \left\langle -25, 25\sqrt{3}, 25\sqrt{2}\right\rangle$

(10) b. Find the position of the object as a vector-valued function of time.

$\boldsymbol{a}(t) = \langle 5,4,-32\rangle$

$\boldsymbol{v}(t) = \langle 5,4,\text{-}32\rangle\ \langle 5t,4t,\text{-}32t\rangle + \left\langle \text{-}25,25\sqrt{3},25\sqrt{2}\right\rangle = \left\langle 5t-25,4t+25\sqrt{3},\text{-}32t+25\sqrt{2}\right\rangle$

$\boldsymbol{s}(t) = \left\langle \frac{5}{2}t^2 - 25t,2t^2+25\sqrt{3}t,\text{-}16t^2+25\sqrt{2}t\right\rangle$

10. Let $\boldsymbol{r}(t) = \left\langle \sqrt{3}t, \sin t, \cos t\right\rangle$.

(10) a. Find the equation of the line that is tangent to the graph of $\boldsymbol{r}$ at the point $t = \frac{\pi}{6}$.

$\boldsymbol{r}'(t) = \left\langle \sqrt{3}, \cos t, \text{-}\sin t\right\rangle$

$\boldsymbol{r}\left(\frac{\pi}{6}\right) = \left\langle \frac{\sqrt{3}\pi}{6},\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle$

$\boldsymbol{r}'\left(\frac{\pi}{6}\right) = \left\langle \sqrt{3},\frac{\sqrt{3}}{2},\text{-}\frac{1}{2}\right\rangle$

$\boldsymbol{l}(t) = \left\langle \frac{\sqrt{3}\pi}{6},\frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle + t\left\langle \sqrt{3},\frac{\sqrt{3}}{2},\text{-}\frac{1}{2}\right\rangle$

(10) b. Find the unit tangent vector $\boldsymbol{T}(t)$.

$\|\boldsymbol{r}'(t)\| = 2$

$\boldsymbol{T}(t) = \frac{1}{2}\left\langle \sqrt{3}, \cos t, \text{-}\sin t\right\rangle$

(10) c. Find the unit normal vector $\boldsymbol{N}(t)$.

$\boldsymbol{T}'(t) = \frac{1}{2}\langle 0,\text{-}\sin t, \text{-}\cos t\rangle$

$\|\boldsymbol{T}'(t)\| = \frac{1}{2}$

$\boldsymbol{N}(t) = \langle 0,\text{-}\sin t, \text{-}\cos t\rangle$

(10) d. Find the curvature $\kappa(t)$.

$\kappa(t) = \dfrac{\frac{1}{2}}{2} = \frac{1}{4}$

(10) e. Find the length of the curve from $t = 0$ to $t = 4$.

$\displaystyle\int_0^4 2\,dt = 2t\Big|_0^4 = 8$

(10) 11. Let $f(x,y) = \sqrt{x^2+y^2}$.

(10) a. Find the domain and range of f.

D: $\mathbb{R}^2$

R: $[0,\infty)$

(10) b. Sketch or describe the graph of f.

Top half of the cone $x^2+y^2 = z^2$.

12. For each of the following, find the limit or show that it does not exist.

(10) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^2}{x+y^2}$

$\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^3+y^2}{x+y^2}$

$\displaystyle= \lim_{x\to 0} \frac{x^3}{x}$

$\displaystyle= \lim_{x\to 0} x^2$

$= 0$

$\displaystyle\lim_{(0,y)\to(0,0)} \frac{x^3+y^2}{x+y^2}$

$\displaystyle= \lim_{y\to 0} \frac{y^2}{y^2}$

$= 1$

DNE

(10) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x+y}$$

$$= \lim_{(x,y)\to(0,0)} \big(x-y\big)$$

$$= 0$$

$$\lim_{y\to 0} \ln|y| = \text{-}\infty$$

$$\lim_{x\to 0} e^x = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{\ln|y|}{e^x} = \text{-}\infty$$

(10) c. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\ln|y|}{e^x}$

13. Let $f(x,y) = x^2e^y + x^2 - 3y^2$.

(10) a. Calculate all first order and second order partial derivatives.

$f_x(x,y) = 2xe^y + 2x = 2x(e^y + 1)$

$f_y(x,y) = x^2e^y - 6y$

$f_{xy}(x,y) = 2xe^y$

$f_{yx}(x,y) = 2xe^y$

$f_{xx}(x,y) = 2e^y + 2$

$f_{yy}(x,y) = x^2e^y - 6$

(10) b. Find the local extrema and saddle points.

$f_x(x,y) = 2x(e^y + 1) = 0$

$x = 0$

$f_y(0,y) = \text{-}6y = 0$

$y = 0$

CP: (0,0)

$D(0,0) = \text{-}24$

Saddle Point: (0,0,0)

(10) 14. Let $f(x,y) = xy - 2x$. Find the absolute extrema of f on the region bounded by the curves $y = \text{-}x^2 + 1$ and $y = x^2 - 1$.

$f_x(x,y) = y - 2$

$f_y(x,y) = x$

CP: (0,2) (not in region)

$f\,(x,\text{-}x^2 + 1) = \text{-}x^3 - x$

Let $g(x) = \text{-}x^3 - x$.

$g'(x) = \text{-}3x^2 - 1$

$\text{-}3x^2 - 1 = 0$

$\text{-}3x^2 = 1$

No solution.

$f(\text{-}1,0) = 2$

$f(1,0) = \text{-}2$

$f\,(x,x^2 - 1) = x^3 - 3x$

Let $g(x) = x^3 - 3x$.

$g'(x) = 3x^2 - 3$

$3x^2 - 3 = 0$

$x = \pm 1$

$f(\text{-}1,0) = 2$

$f(1,0) = \text{-}2$

Min: -2

Max: 2

Section 5.4: Exam 3

Exam 3 Math 2673 Fall 2018

Name:__________________________

15. Let $f(x,y,z) = 3xy + y^2e^z$

(10) a. Give the linear approximation of f at the point (1,2,0) and use it to approximate $f(.99,2.1,.001)$.

$f_x(x,y,z) = 3y$

$f_y(x,y,z) = 3x + 2ye^z$

$f_z(x,y,z) = y^2e^z$

$f_x(1,2,0) = 6$

$f_y(1,2,0) = 7$

$f_z(1,2,0) = 4$

$f(1,2,0) = 10$

$L(x,y,z) = 10 + 6(x-1) + 7(y-2) + 4z$

$L(.99,2.1,.001) = 10 + 6(\text{-}.01) + 7(.1) + 4(.001) = 10.644$

(10) b. Given that f is differentiable, find $f_{(a,b,c)}(x,y,z)$ for all $(a,b,c) \in \mathbb{R}^3$.

$f_x(a,b,c) = 3b$

$f_y(a,b,c) = 3a + 2be^c$

$f_z(a,b,c) = b^2e^c$

$f'_{(a,b,c)}(x,y,z) = 3bx + (3a + 2be^c)y + (b^2e^c)z$

(10) c. Find the gradient of f.

$\nabla f(x,y,z) = \langle 3y, 3x + 2ye^z, y^2e^z \rangle$

(10) d. Find the directional derivative of f at (1,2,0) in the direction of $\boldsymbol{u} = \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle$.

$\nabla f(1,2,0) = \langle 6,7,4 \rangle$

$\langle 6,7,4 \rangle \cdot \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle$

$= 3 + \frac{7}{2} + 2\sqrt{2}$

$= \frac{13+4\sqrt{2}}{2}$

(10) e. In what direction does the maximum rate of change of f occur at (1,2,0)? What is the maximum rate of change?

$\nabla f(1{,}2{,}0) = \langle 6{,}7{,}4 \rangle$

$\|\nabla f(1{,}2{,}0)\| = \| \langle 6{,}7{,}4 \rangle \| = \sqrt{101}$

(10) f. Find an equation for the tangent plane to the level surface $f(x{,}y{,}z) = 10$ at the point (1,2,0).

$\nabla f(1{,}2{,}0) = \langle 6{,}7{,}4 \rangle$

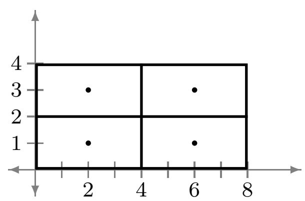
$6(x-1) + 7(y-2) + 4z = 0$

(10) g. Find an equation for the normal line to the level surface $f(x{,}y{,}z) = 10$ at the point (1,2,0).

$\boldsymbol{l}(t) = \langle 1{,}2{,}0 \rangle + t\langle 6{,}7{,}4 \rangle$

16. Let $R = [0{,}8] \times [0{,}4]$.

(10) a. Use the midpoint rule with $m = n = 2$ to estimate $\displaystyle\iint_R x^2 e^y \, dA$.



$8f(2{,}1) + 8f(6{,}1) + 8f(2{,}3) + 8f(6{,}3) = 8(4e) + 8(36e) + 8(4e^3) + 8(36e^3) = 320e + 320e^3$

(10) b. Evaluate $\displaystyle\iint_R x^2 e^y \, dA$.

$$\iint_R x^2 e^y \, dA$$

$$= \int_0^8 \int_0^4 x^2 e^y \, dydx$$

$$= \int_0^8 x^2 e^y \Big|_0^4 \, dx$$

$$= \int_0^8 x^2(e^4 - 1) \, dx$$

$$= \tfrac{1}{3}(e^4 - 1)x^3 \Big|_0^8$$

$$= \tfrac{512}{3}(e^4 - 1)$$

$$\iint_R x^2 e^y \, dA$$

$$= \int_0^4 \int_0^8 x^2 e^y \, dxdy$$

$$= \int_0^4 \tfrac{1}{3}x^3 e^y \Big|_0^8 \, dy$$

$$= \int_0^4 \tfrac{512}{3} e^y \, dy$$

$$= \tfrac{512}{3} e^y \Big|_0^4$$

$$= \tfrac{512}{3}(e^4 - 1)$$

(10) 17. Maximize and minimize $f(x,y,z) = xz + yz$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Let $g(x,yz,) = x^2 + y^2 + z^2 - 1$.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle z,z,x + y\rangle = \lambda \langle 2x,2y,2z\rangle$

If $\lambda = 0$, then $z = 0$ and $y = \text{-}x$.

$f(x,y,0) = 0$

Now suppose that $\lambda \neq 0$.

From $\langle z,z,x + y\rangle = \lambda \langle 2x,2y,2z\rangle$ we have the following.

$z = 2\lambda x$ $\qquad$ $z = 2\lambda y$ $\qquad$ $x + y = 2\lambda z$

From the first two equations, we see that $x = y$.

If $x = 0$, then $y = 0$ and $z = 0$. This is not possible since $x^2 + y^2 + z^2 = 1$. So $x \neq 0$. Also, from the first equation we now have $\lambda = \frac{z}{2x}$.

Considering the third equation, yields the following.

$x + y = 2\lambda z$

$2x = 2\lambda z$

$x = \lambda z$

$\lambda = \frac{x}{z}$

So now we have the following.

$\frac{z}{2x} = \frac{x}{z}$

$2x^2 = z^2$

$x^2 + y^2 + z^2 = 1$

$2x^2 + z^2 = 1$

$2z^2 = 1$

$z = \pm \frac{1}{\sqrt{2}}$

$x = \pm \frac{1}{2}$

$y = \pm \frac{1}{2}$

$f\left(\text{-}\frac{1}{2},\text{-}\frac{1}{2}\text{-}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ (max)

$f\left(\text{-}\frac{1}{2},\text{-}\frac{1}{2}\frac{1}{\sqrt{2}}\right) = \text{-}\frac{1}{\sqrt{2}}$ (min)

$f\left(\frac{1}{2},\frac{1}{2}\text{-}\frac{1}{\sqrt{2}}\right) = \text{-}\frac{1}{\sqrt{2}}$ (min)

$f\left(\frac{1}{2},\frac{1}{2}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ (max)

Section 5.5: Final

Final Exam Math 2673 Fall 2018

(10) 18. Give the equation of the sphere with center (2,-1,5) and tangent plane $x = \text{-}3$.

(10) 19. A 10 lb ornament is hung from a ceiling with two wires. One wire makes an angle of $30°$ with the ceiling. The other wire makes an angle of $60°$ with the ceiling. Find the tension on each wire. Hint: See example 37.

(10) 20. Find the volume of the parallelepiped determined by the vectors $\boldsymbol{u} = \langle 1,\text{-}2,1\rangle$, $\boldsymbol{v} = \langle 0,\text{-}1,3\rangle$, and $\boldsymbol{w} = \langle 2,2,0\rangle$.

21. Sketch or describe each of the following.

(10) a. $\frac{x^2}{4} - \frac{y^2}{9} = z^2$

(10) b. $\frac{x^2}{4} + \frac{y^2}{9} = z^2 + 1$

(10) c. $z - x^2 = y^2$

22. Consider the vector function $\boldsymbol{r}(t) = \langle \sin t, \cos t, t \rangle$.

(10) a. Give the equation(s) of the line that is tangent to the graph of $\boldsymbol{r}$ at the point (0,-1,π).

(10) b. Give the length of the curve from the point (0,1,0) to the point (0,-1,π).

23. For each of the following, find the limit or prove that it does not exist.

(10) a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x - y}}$

(10) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x + y^2}$

(10) 24. Find all local extrema and saddle points of the following function.

$f(x,y) = x^2y + xy$

(10) 25. Maximize and minimize $f(x,y,z) = x^2 + y - z$ subject to the constraints $x^2 + y^2 = 1$ and $y + z = 0$.

(10) 26. Let R be the region in the plane bounded by the curves $y = 0$, $x = 1$, and $y = \sqrt{x}$. Calculate the following integral.

$$\iint\limits_R \frac{e^y}{\sqrt{x}}\, dA$$

(10) 27. Evaluate the following integral.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$$

(10) 28. Let R be the region bounded by the planes $x = 0$, $x = 1$, $y = 0$, $z = 0$, and $z = x + y$.

Calculate the following integral.

$$\iiint\limits_R x\, dV$$

(10) 29. Let R be the region bounded by the top half of the cone $x^2+y^2=z^2$ and the plane $z=1$. Express $\displaystyle\iiint_R x\,dV$ in terms of rectangular, cylindrical, and spherical coordinates. Evaluate one of the integrals.

Chapter 6: Fall 2020

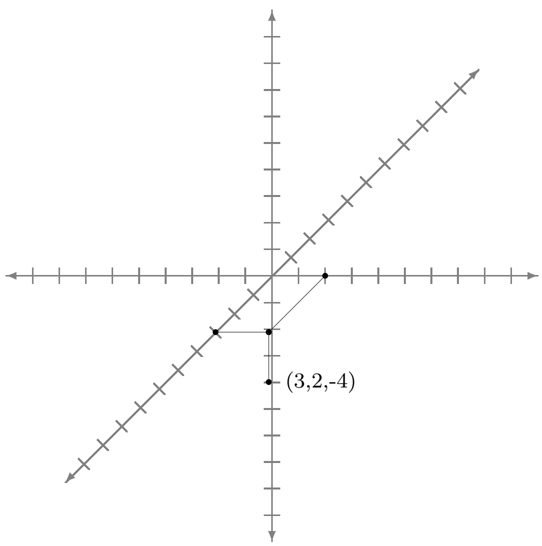
Section 6.1: Quizzes

Quiz 60 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Plot the following point.

(3,2,-4)



(2) 2. Give the center and radius of each of the following spheres. Do the spheres intersect?

$x^2 + y^2 + z^2 = 1$

Center: (0,0,0)

Radius: 1

$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$

Center: (1,1,1)

Radius: 1

Since the distance between the centers of the spheres is $\sqrt{3}$ and $\sqrt{3} < 1 + 1$, the spheres do intersect.

(1) 3. Find a unit vector parallel to ⟨-1,4⟩.

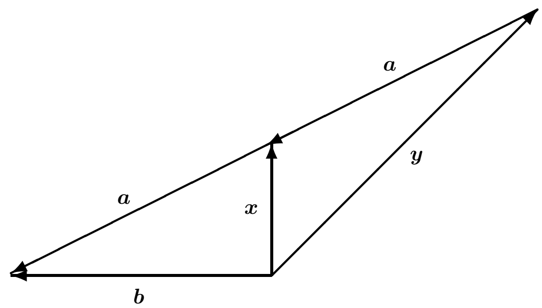
$\|\langle\text{-}1,4\rangle\| = \sqrt{17}$

$\frac{1}{\sqrt{17}}\langle\text{-}1,4\rangle$

(2) 4. Let θ be the measure of the angle between the vectors ⟨1,7⟩ and ⟨3,5⟩. Calculate $\cos\theta$.

$\cos\theta = \frac{\langle 1,7\rangle \cdot \langle 3,5\rangle}{\|\langle 1,7\rangle\|\|\langle 3,5\rangle\|} = \frac{38}{\sqrt{50}\sqrt{34}} = \frac{19}{5\sqrt{17}}$

(2) 5. Consider the figure below. Express $\boldsymbol{x}$ and $\boldsymbol{y}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



$\boldsymbol{x} + \boldsymbol{a} = \boldsymbol{b}$

$\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{a}$

$\boldsymbol{y} + \boldsymbol{a} = \boldsymbol{x}$

$\boldsymbol{y} + \boldsymbol{a} = \boldsymbol{b} - \boldsymbol{a}$

$\boldsymbol{y} = \boldsymbol{b} - 2\boldsymbol{a}$

6. Find α such that the vectors $\langle 1,\text{-}2,5\rangle$ and $\langle \text{-}2,4,\alpha\rangle$ are

(1) a. parallel

$\text{-}2\langle 1,\text{-}2,5\rangle = \langle \text{-}2,4,\alpha\rangle$

$\langle \text{-}2,4,\text{-}10\rangle = \langle \text{-}2,4,\alpha\rangle$

$\alpha = \text{-}10$

(1) b. perpendicular

$\langle 1,\text{-}2,5\rangle \cdot \langle \text{-}2,4,\alpha\rangle = 0$

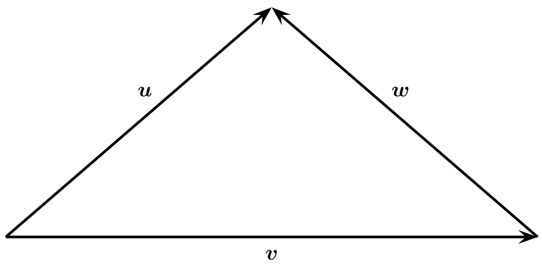
$\text{-}2 - 8 + 5\alpha = 0$

$\alpha = 2$

Quiz 61 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. In the figure below, $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are unit vectors. Calculate $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{v} \cdot \boldsymbol{w}$.



a. $\boldsymbol{u} \cdot \boldsymbol{v}$

Since the angle between the vectors is $\frac{\pi}{3}$ and the vectors are unit vectors, $\boldsymbol{u} \cdot \boldsymbol{v} = \cos\frac{\pi}{3} = \frac{1}{2}$.

b. $\boldsymbol{v} \cdot \boldsymbol{w}$

Since the angle between the vectors is $\frac{2\pi}{3}$ and the vectors are unit vectors, $\boldsymbol{u} \cdot \boldsymbol{v} = \cos\frac{2\pi}{3} = -\frac{1}{2}$.

(2) 2. Find the work done by a force of $\boldsymbol{F} = \langle 2,3,-5\rangle$ that moves an object from the point (-1,4,7) to the point (4,5,2). Force is measured in pounds and distance in feet.

Set $\boldsymbol{D} = \langle 5,1,-5\rangle$. Then the work done is $\boldsymbol{F} \cdot \boldsymbol{D}$ ft-lb = 38 ft-lb.

3. Set $\boldsymbol{v} = \langle 1,0,3\rangle$ and $\boldsymbol{w} = \langle 2,-1,6\rangle$. Calculate each of the following.

(2) a. $\boldsymbol{v} \times \boldsymbol{w}$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 3 \\ 2 & -1 & 6 \end{vmatrix} = \langle 3,0,-1\rangle$$

(1) b. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w}$

$$= \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|}$$

$$= \frac{20}{\sqrt{10}}$$

$$= 2\sqrt{10}$$

(1) c. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} =$

$$\left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|}\right)\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$$

$$= 2\boldsymbol{v}$$

$$= 2\langle 1,0,3\rangle$$

(2) 4. Give the equation of the plane containing the points (1,-2,7), (1,4,-3), (2,0,6).

Set $\boldsymbol{v} = \langle 1-1, 4--2, -3-7\rangle = \langle 0,6,-10\rangle = 2\langle 0,3,-5\rangle$ and $\boldsymbol{w} = \langle 2-1, 0--2, 6-7\rangle = \langle 1,2,-1\rangle$. Then a normal vector to the plane is $\frac{1}{2}\boldsymbol{v}\times\boldsymbol{w}$.

$$\frac{1}{2}\boldsymbol{v}\times\boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 3 & -5 \\ 1 & 2 & -1 \end{vmatrix} = \langle 14,-5,-3\rangle.$$

$$14(x-1) - 5(y+2) - 3(z-7) = 0$$

(1) 5. Give the equation of the plane that contains the set of all points that are equidistant from the points (1,-1,2) and (3,0,1).

Solution 1:

The midpoint of the line segment joining the two points is $\left(2,-\frac{1}{2},\frac{3}{2}\right)$. The vector $\langle 3-1, 0--1, 1-2\rangle = \langle 2,1,-1\rangle$ is orthogonal to the plane. The equation of the plane containing the point $\left(2,-\frac{1}{2},\frac{3}{2}\right)$ with normal vector $\langle 2,1,-1\rangle$ is $2(x-2) + \left(y+\frac{1}{2}\right) - \left(z-\frac{3}{2}\right) = 0$.

Solution 2:

If a point (x,y,z) is equidistant from (1,-1,2) and (3,0,1), then we have the following.

$$\sqrt{(x-1)^2 + (y+1)^2 + (z-2)^2} = \sqrt{(x-3)^2 + (y-0)^2 + (z-1)^2}$$

$$(x-1)^2 + (y+1)^2 + (z-2)^2 = (x-3)^2 + (y-0)^2 + (z-1)^2$$

$$x^2 - 2x + 1 + y^2 + 2y + 1 + z^2 - 4z + 4 = x^2 - 6x + 9 + y^2 + z^2 - 2z + 1$$

$$4x + 2y - 2z = 4$$

$$2x + y - z = 2$$

Quiz 62 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Give the equation of the line containing the points (2,-1,3) and (3,-2,7).

$\boldsymbol{v} = \langle 1,-1,4\rangle$

Parametric	Symmetric	Vector
$x = 2 + t$	$x - 2 = -y - 1 = \frac{z-3}{4}$	$\boldsymbol{l}(t) = \langle 2,-1,3\rangle + t\langle 1,-1,4\rangle$
$y = -1 - t$		
$z = 3 + 4t$		

(2) 2. Find an equation of the line where the following planes intersect.

$2x - y + 3z = 1$

$-3x + y + 5z = 7$

Let $z = 0$.

$$\begin{array}{rcrcl} 2x & - & y & = & 1 \\ -3x & + & y & = & 7 \end{array}$$

$-x = 8$

$x = -8$

$y = -17$

So the point (-8,-17,0) is on the line of intersection of the two planes.

Normal vectors of the two planes are $\boldsymbol{n}_1 = \langle 2,-1,3\rangle$ and $\boldsymbol{n}_2 = \langle -3,1,5\rangle$. So a vector parallel to the line of intersection is $\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle -8,-19,-1\rangle$. Equations of the line passing through the point (-8,-17,0) with parallel vector $\langle -8,-19,-1\rangle$ are

$x = -8 - 8t$

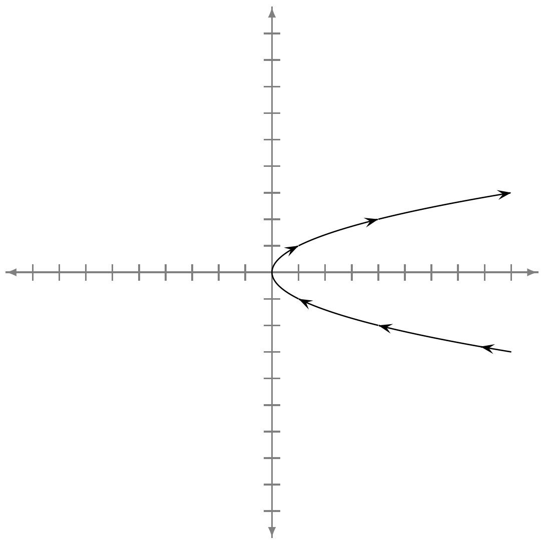
$y = -17 - 19t$

$z = -t.$

(2) 3. Sketch the curve of the given vector equation. Draw arrows along the curve in the direction of increasing t.

$\boldsymbol{f}(t) = \langle t^2, t \rangle$

$x = y^2$



4. Consider the vector-valued function $\boldsymbol{f}(t) = \langle t\cos t, t\sin t, t^2 \rangle$.

(1) a. Show that the graph of the function lies on the surface $z = x^2 + y^2$.

$x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2 = z$

(1) b. Identify the surface $z = x^2 + y^2$.

Circular paraboloid.

(1) c. Sketch the graph of $\boldsymbol{f}$.

Quiz 63 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{r}(t) = \langle t^2+1, \cos \pi t, \ln t \rangle$.

(3) a. Calculate $\boldsymbol{r}'(t)$.

$\boldsymbol{r}'(t) = \left\langle 2t, \text{-}\pi \sin \pi t, \frac{1}{t} \right\rangle$

(2) b. Give the equation of the line that is tangent to the graph of $\boldsymbol{r}$ at the point (2,-1,0).

$\boldsymbol{r}(1) = (2,\text{-}1,0)$

$\boldsymbol{r}'(1) = (2,0,1)$

$\boldsymbol{l}(t) = \langle 2,\text{-}1,0 \rangle + t\langle 2,0,1 \rangle$

(3) 2. Calculate $\displaystyle\int \langle t^3, \cos t, \sec^2 t \rangle \, dt = \left\langle \tfrac{1}{4}t^4, \sin t, \tan t \right\rangle + \boldsymbol{C}$

(1) 3. Calculate the following limit. Hint: Recall L'Hôpital's Rule.

$\lim_{t\to 0} \langle \frac{1-\cos t}{t}, t \ln t, \sin t \rangle = \langle 1,0,0 \rangle$

(1) 4. Give an equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$.

First, find an equation for the curve. From the equations above, we have $y = \sqrt{1-x^2}$ and $z = \sqrt{1-x^2}$. So $\boldsymbol{r}(t) = \left\langle t, \sqrt{1-t^2}, \sqrt{1-t^2} \right\rangle$. Note that $\boldsymbol{r}\left(\frac{1}{2}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\rangle$. Alternatively, an equation of the curve is $\boldsymbol{r}(\theta) = \langle \cos\theta, \sin\theta, \sin\theta \rangle$. Note that $\boldsymbol{r}\left(\frac{\pi}{3}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

$\boldsymbol{r}'(\theta) = \langle \text{-}\sin\theta, \cos\theta, \cos\theta \rangle$

$\boldsymbol{r}'\left(\frac{\pi}{3}\right) = \left\langle \text{-}\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$

$\boldsymbol{l}(t) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\rangle + t\left\langle \text{-}\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$

Quiz 64 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. The motion of an object is described by $\boldsymbol{r}(t) = \left\langle 5\cos t, \sqrt{11}t, 5\sin t \right\rangle$.

(2) a. Find the unit tangent vector $\boldsymbol{T}(t)$.

$\boldsymbol{r}'(t) = \left\langle -5\sin t, \sqrt{11}, 5\cos t \right\rangle$.

$\|\boldsymbol{r}'(t)\| = 6$

$\boldsymbol{T}(t) = \frac{1}{6}\left\langle -5\sin t, \sqrt{11}, 5\cos t \right\rangle$

(1) b. Find the unit normal vector $\boldsymbol{N}(t)$.

$\boldsymbol{T}'(t) = \frac{1}{6}\langle -5\cos t, 0, -5\sin t\rangle$

$\|\boldsymbol{T}'(t)\| = \frac{5}{6}$

$\boldsymbol{N}(t) = \langle -\cos t, 0, -\sin t\rangle$

(1) c. Find the binormal vector $\boldsymbol{B}(t)$.

$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t) = \frac{1}{6}\left\langle -\sqrt{11}\sin t, -5, \sqrt{11}\cos t \right\rangle$

(1) d. Find the curvature $\kappa(t)$.

$\frac{\|\boldsymbol{T}'(t)\|}{\|\boldsymbol{r}'(t)\|} = \frac{5}{36}$

(1) e. Find the arc length of the curve from $t = 0$ to $t = 4$.

$$\int_0^4 \|\boldsymbol{r}'(t)\|\,dt = \int_0^4 6\,dt = 6t\Big|_0^4 = 24$$

(1) f. Reparameterize the curve with respect to arc length beginning at the point $t = 0$.

$$s = \int_0^t \|\boldsymbol{r}'(q)\|\,dq = \int_0^t 6\,dq = 6q\Big|_0^t = 6t$$

$t = \frac{1}{6}s$

$\boldsymbol{r}\left(\frac{s}{6}\right) = \left\langle 5\cos\left(\frac{s}{6}\right), \frac{\sqrt{11}s}{6}, 5\sin\left(\frac{s}{6}\right) \right\rangle$

(1) g. Find the velocity of the object when $t = \frac{\pi}{6}$.

$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = \left\langle -5\sin t, \sqrt{11}, 5\cos t \right\rangle$

$\boldsymbol{v}\left(\frac{\pi}{6}\right) = \left\langle -\frac{5}{2}, \sqrt{11}, \frac{5\sqrt{3}}{2} \right\rangle$

(1) h. Find the acceleration of the object when $t = \frac{\pi}{6}$.

$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \langle -5\cos t, 0, -5\sin t\rangle$

$\boldsymbol{a}\left(\frac{\pi}{6}\right) = -5\left\langle \frac{\sqrt{3}}{2}, 0, \frac{1}{2} \right\rangle$

(1) i. Find the speed of the object when $t = \frac{\pi}{6}$.

$v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\| = 6$

$v\left(\frac{\pi}{6}\right) = 6$

Quiz 65 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. For the function below, give the domain and range and sketch the graph.

$f(x,y) = \sqrt{x^2 - y^2}$

D: $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 \geq 0\} = \{(x,y) \in \mathbb{R}^2 : -|x| \leq |y| \leq |x|\}$

The graph is the top half of the cone $x^2 = y^2 + z^2$.

2. Calculate the following limits.

(3) a. $\displaystyle\lim_{(x,y)\to(1,3)} \frac{x^2y + x}{x^3 - 2} = -4$

(3) b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y + x^2y^2}{x^4 + y^4}$

$$\lim_{(x,0)\to(0,0)} \frac{x^3y + x^2y^2}{x^4 + y^4} = \lim_{y\to 0} \frac{0}{y^4} = 0$$

$$\lim_{(x,x)\to(0,0)} \frac{x^3y + x^2y^2}{x^4 + y^4} = \lim_{x\to 0} \frac{2x^4}{2x^4} = 1$$

So the limit does not exist.

(2) c. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy - y}{\sqrt{x} - 1} = \lim_{(x,y)\to(1,1)} \frac{y(x-1)}{\sqrt{x} - 1} = \lim_{(x,y)\to(1,1)} y(\sqrt{x} + 1) = 2$

(1) d. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2 + y^2}} = 0$

Proof: To see that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2 + y^2}} = 0$, let $\varepsilon > 0$ be given. Set $\delta = \varepsilon$. Now suppose that $d[(x,y),(0,0,)] < \delta$.

Then

$$\sqrt{(x-0)^2 + (y-0)^2} < \delta$$

$$\sqrt{x^2 + y^2} < \delta$$

$$\sqrt{y^2} \leq \sqrt{x^2 + y^2} < \delta$$

$$|y| \leq \sqrt{x^2 + y^2} < \delta$$

$$|y| \cdot |y| \leq |y|\sqrt{x^2 + y^2}$$

$$y^2 < \delta\sqrt{x^2 + y^2} \qquad \text{(Since } |y| < \delta.\text{)}$$

$$\frac{y^2}{\sqrt{x^2 + y^2}} < \delta$$

$$\left|\frac{y^2}{\sqrt{x^2 + y^2}} - 0\right| < \delta = \varepsilon$$

as desired. ■

Quiz 66 Name:________________________

<u>Directions:</u> Show all of your work and justify all of your answers.

1. Set $f(x,y) = xy^2 - x^2y$.

(6) a. Compute each of the following partial derivatives.

i. $f_x(x,y) = y^2 - 2xy = y(y - 2x)$

ii. $f_y(x,y) = 2xy - x^2 = x(2y - x)$

iii. $f_{xx}(x,y) = -2y$

iv. $f_{yy}(x,y) = 2x$

v. $f_{xy}(x,y) = 2y - 2x$

vi. $f_{yx}(x,y) = 2y - 2x$

(1) b. Find the critical points of f.

The only critical point is (0,0).

If $f_x(x,y) = y(y - 2x) = 0$, then $y = 0$ or $y = 2x$. The only solution to $f_y(x,0) = 0$ is 0. So (0,0) is a critical point. Also, the only solution to $f_y(x,2x) = 0$ is 0. So (0,0) is the only critical point.

(1) c. Apply the Second Partials Test to the Critical Points. Find all local extrema. List the critical points for which the Second Partials Test is inconclusive.

Since $f_{xx}(0,0) = f_{yy}(0,0) = f_{xy}(0,0) = 0$, $D(0,0) = 0$. So the Second Partials Test is inconclusive.

(2) d. Let T be the region bounded by the triangle with vertices (0,0), (1,0), and (1,1). Find the maximum and minimum values of f on T.

Recall that the only critical point of f is (0,0) which is also on the boundary of T. So the extreme values of f on T will occur on the boundary of T.

Case 4: $y = 0$ and $0 \leq x \leq 1$

$f(x,0) = 0$

Case 5: $x = 1$ and $0 \leq y \leq 1$

Note $f(1,y) = y^2 - y = y(y - 1)$ which is a parabola with vertex $\left(\frac{1}{2},-\frac{1}{4}\right)$. So the minimum value of f on the line segment is $f\left(1,\frac{1}{2}\right) = -\frac{1}{4}$ and the maximum value of f on the line segment is $f(1,0) = f(1,1) = 0$.

Case 6: $y = x$ and $0 \leq x \leq 1$

$f(x,x) = 0$

So the maximum value of f on T is 0 and occurs at every point on the boundary where $y = 0$ or $y = x$ and the minimum value of f on T is $-\frac{1}{4}$ and occurs at the point $\left(1,\frac{1}{2}\right)$.

Quiz 67 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $f(x,y) = y\cos x + e^{xy} + y$.

(3) a. Give the equation of the plane that is tangent to the surface $z = f(x,y)$ at the point (0,1,3).

$f_x(x,y) = \text{-}y\sin x + ye^{xy}$

$f_y(x,y) = \cos x + xe^{xy} + 1$

$f_x(0,1) = 1$

$f_y(0,1) = 2$

$z - 3 = 1(x-0) + 2(y-1)$

$z = x + 2y + 1$

(2) b. Give the linear approximation of f at the point (0,1) and use it to approximate $f(0.01,0.99)$.

$L(x,y) = x + 2y + 1$

$L(0.01,0.99) = .01 + 1.98 + 1 = 2.99$

2. Consider the function $z = f(x,y) = x^3y^2$ where $x = 2s + t^3$ and $y = \sin s + t$. Calculate each of the following.

(1) a. $\frac{\partial z}{\partial s}$

$\frac{\partial z}{\partial x} = 3x^2y^2$

$\frac{\partial z}{\partial y} = 2x^3y$

$\frac{\partial x}{\partial s} = 2$

$\frac{\partial x}{\partial t} = 3t^2$

$\frac{\partial y}{\partial s} = \cos s$

$\frac{\partial y}{\partial t} = 1$

$\frac{\partial z}{\partial s}$

(1) b. $\frac{\partial z}{\partial t}$

$= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$

$= (3x^2y^2)(2) + 2x^3y\cos s$

$= 6(2s+t^3)^2(\sin s+t)^2+2(2s+t^3)^3(\sin s+t)\cos s$

$\frac{\partial z}{\partial t}$

$= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

$= (3x^2y^2)(3t^2) + (2x^3y)(1)$

$= 9t^2(2s + t^3)^2(\sin s + t)^2 + 2(2s + t^3)^3(\sin s + t)$

3. Set $f(x,y) = x^2 + y - 1$.

(2) a. Show that f is differentiable at (a,b) for all $(a,b) \in \mathbb{R}^2$.

$f_x(x,y) = 2x$

$f_y(x,y) = 1$

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - \left[f_x(a,b)\cdot h + f_y(a,b)\cdot k\right]}{\|(h,k)\|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{(a+h)^2 + (b+k) - 1 - (a^2 + b - 1) - (2ah + k)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{a^2 + 2ah + h^2 + b + k - 1 - a^2 - b + 1 - 2ah - k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2+k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2}{|h|}$$

$$= \lim_{(h,k)\to(0,0)} |h|$$

$$= 0$$

(1) b. Give $f'_{(a,b)}(x,y)$.

$f'_{(a,b)}(x,y) = \nabla f(a,b) \cdot \langle x,y\rangle = \langle 2a,1\rangle \cdot \langle x,y\rangle = 2ax + y$

Quiz 68 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $f(x,y,z) = e^{xy} + y^2 + xz$.

(3) a. Calculate the gradient of f at (1,-1,2), $\nabla f(1,\text{-}1,2)$.

$\nabla f(x,y,z) = \langle ye^{xy} + z, xe^{xy} + 2y, x\rangle$

$\nabla f(1,\text{-}1,2) = \left\langle 2 - \frac{1}{e}, \text{-}2 + \frac{1}{e}, 1\right\rangle$

(2) b. Give the directional derivative of f at the point(1,-1,2) in the direction of $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right\rangle$.

$\left\langle 2 - \frac{1}{e}, \text{-}2 + \frac{1}{e}, 1\right\rangle \cdot \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right\rangle = \frac{1}{\sqrt{2}}$

(2) c. Give the equation of the tangent plane to the level surface $f(x,y,z) = 3 + \frac{1}{e}$ at the point (1,-1,2).

$(2 - \frac{1}{e})(x - 1) + (\text{-}2 + \frac{1}{e})(y + 1) + 1(z - 2) = 0$

$\left\langle 2 - \frac{1}{e}, \text{-}2 + \frac{1}{e}, 1\right\rangle$

(1) d. Give the equation of the normal line to the level surface $f(x,y,z) = 3 + \frac{1}{e}$ at the point (1,-1,2).

$\boldsymbol{l}(t) = \langle 1,\text{-}1,2\rangle + t\left\langle 2 - \frac{1}{e}, \text{-}2 + \frac{1}{e}, 1\right\rangle$

(1) 2. Find the points on the unit sphere that are nearest to and farthest from the point (1,1,1).

Set $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$, $g(x,y,z) = x^2 + y^2 + z^2$, and find the maximum and minimum values of f subject to the constraint $g(x,y,z) = 1$. Use Lagrange multipliers.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle 2(x-1), 2(y-1), 2(z-1)\rangle = \lambda\langle 2x, 2y, 2z\rangle$

$x - 1 = \lambda x$

$x(1-\lambda) = 1$

$x = \frac{1}{1-\lambda}$

$y - 1 = \lambda y$

$y(1-\lambda) = 1$

$y = \frac{1}{1-\lambda}$

$z - 1 = \lambda z$

$z(1-\lambda) = 1$

$z = \frac{1}{1-\lambda}$

$x = y = z$

$x^2 + y^2 + z^2 = 1$

$3x^2 = 1$

$x = \pm\sqrt{\frac{1}{3}}$

$\left(\frac{1}{1-\lambda}\right)^2 = \frac{1}{3}$

$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = (1 - \sqrt{3})^2$

$f\left(\frac{\text{-}1}{\sqrt{3}}, \frac{\text{-}1}{\sqrt{3}}, \frac{\text{-}1}{\sqrt{3}}\right) = (1 + \sqrt{3})^2$

(2) 3. Calculate $\displaystyle\iint\limits_R x\sin(xy)\,dA$ where $R = \left[0,\frac{\pi}{2}\right] \times [0,1]$.

$$\iint\limits_R x\sin(xy)\,dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 x\sin(xy)\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}} \text{-}\cos(xy)\Big|_0^1\,dx$$

$$= \int_0^{\frac{\pi}{2}} \big(\text{-}\cos x - \text{-}1\big)dx$$

$$= \int_0^{\frac{\pi}{2}} \big(1 - \cos x\big)dx$$

$$= \big(x - \sin x\big)\Big|_0^{\frac{\pi}{2}}$$

$$= \tfrac{\pi}{2} - 1$$

Quiz 69 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Evaluate the following iterated integral.

$$\int_0^{\frac{\pi}{2}}\int_0^{\sin x}(\cos x)e^y\,dy\,dx = \int_0^{\frac{\pi}{2}}(\cos x)e^y\Big|_0^{\sin x}\,dx = \int_0^{\frac{\pi}{2}}\big[(\cos x)e^{\sin x}-\cos x\big]dx = \left(e^{\sin x}-\sin x\right)\Big|_0^{\frac{\pi}{2}}$$
$$= e-2$$

(4) 2. The region R is bounded by the curves $x=0$, $y=1$, and $y=\sqrt{x}$. Give $\displaystyle\iint\limits_R \frac{1}{y^3+1}\,dA$ as iterated integrals for both orders of integration. Evaluate one of the iterated integrals.

$$\int_0^1\int_{\sqrt{x}}^1 \frac{1}{y^3+1}\,dy\,dx$$

$$\int_0^1\int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$

$$\int_0^1\int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy = \int_0^1 \frac{x}{y^3+1}\Big|_0^{y^2}\,dy = \int_0^1\frac{y^2}{y^3+1}\,dy = \ln|y^3+1|\Big|_0^1 = \ln 2$$

(3) 3. Evaluate the following iterated integral by converting to polar coordinates.

$$\int_0^1\int_0^{\sqrt{1-x^2}} xy^2\,dy\,dx$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 (r\cos\theta)(r^2\sin^2\theta)r\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\sin^2\theta\,dr\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\left(\tfrac{1}{5}r^5\cos\theta\sin^2\theta\right)\Big|_0^1\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\tfrac{1}{5}\cos\theta\sin^2\theta\,d\theta$$

$$=\tfrac{1}{15}\sin^3\theta\Big|_0^{\frac{\pi}{2}}$$

$$=\tfrac{1}{15}$$

Quiz 70 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Set T to be the solid tetrahedron bounded by the coordinate planes and the plane $x+y+z=1$. Calculate the following integral.

$$\iiint_T e^z\,dV$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} e^z\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} e^z\Big|_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left(e^{1-x-y}-1\right)dy\,dx$$

$$= \int_0^1 \left(\text{-}e^{1-x-y}-y\right)\Big|_0^{1-x}\,dx$$

$$= \int_0^1 \left[\text{-}1-(1-x)-(\text{-}e^{1-x}-0)\right]dx$$

$$= \int_0^1 \left(e^{1-x}+x-2\right)dx$$

$$= \left(\text{-}e^{1-x}+\tfrac{1}{2}x^2-2x\right)\Big|_0^1$$

$$= \text{-}1+\tfrac{1}{2}-2-(\text{-}e+0-0)$$

$$= e-\tfrac{5}{2}$$

2. Set R to be the region bounded by the half cone $z=\sqrt{x^2+y^2}$ and the plane $z=1$.

(2) a. Give the integral $\iiint_R z^2\,dV$ as an iterated integral using rectangular coordinates. Do not evaluate the integral.

$$\iiint_R z^2\,dV = \int_{\text{-}1}^1\int_{\text{-}\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 z^2\,dz\,dy\,dx$$

(3) b. Give the integral $\iiint\limits_R z^2\,dV$ as an iterated integral using cylindrical coordinates and evaluate the integral.

$$\iiint\limits_R z^2\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 r z^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \tfrac{1}{3} r z^3\Big|_r^1\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \tfrac{1}{3} r(1-r^3)\,dr\,d\theta$$

$$= \tfrac{1}{3}\int_0^{2\pi}\int_0^1 \left(r - r^4\right)dr\,d\theta$$

$$= \tfrac{1}{3}\int_0^{2\pi} \left(\tfrac{1}{2}r^2 - \tfrac{1}{5}r^5\right)\Big|_0^1\,d\theta$$

$$= \tfrac{1}{3}\int_0^{2\pi} \tfrac{3}{10}\,d\theta$$

$$= \tfrac{1}{10}\theta\Big|_0^{2\pi}$$

$$= \tfrac{\pi}{5}$$

(3) c. Give the integral $\iiint\limits_R z^2\,dV$ as an iterated integral using spherical coordinates and evaluate the integral.

$$\iiint\limits_R z^2\,dV$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sec\phi} (\rho^2\cos^2\phi)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \tfrac{1}{5}\rho^5\sin\phi\cos^2\phi\Big|_0^{\sec\phi}\,d\theta\,d\phi$$

$$= \tfrac{1}{5}\int_0^{\frac{\pi}{4}}\int_0^{2\pi} \sec^3\phi\sin\phi\,d\theta\,d\phi$$

$$= \tfrac{1}{5}\int_0^{\frac{\pi}{4}}\int_0^{2\pi} (\cos\phi)^{-3}\sin\phi\,d\theta\,d\phi$$

$$= \tfrac{1}{5}\int_0^{\frac{\pi}{4}} \theta\Big|_0^{2\pi} (\cos\phi)^{-3}\,\phi\sin\phi\,d\phi$$

$$= \tfrac{2\pi}{5}\int_0^{\frac{\pi}{4}} (\cos\phi)^{-3}\sin\phi\,d\phi$$

$$= \tfrac{2\pi}{5}\left[\tfrac{1}{2}(\cos\phi)^{-2}\right]\Big|_0^{\frac{\pi}{4}}$$

$$= \tfrac{\pi}{5}$$

Quiz 71 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Suppose that $f:\mathbb{R}^3 \to \mathbb{R}$, $\nabla f(x,y,z) = \langle ye^x + xye^x, xe^x, y\cos z\rangle$, and $f(1,1,0) = e$. Find $f(x,yz)$.

$f(x,yz) = xye^x + y\sin z$

(2) 2. Suppose that $f:\mathbb{R}^3 \to \mathbb{R}$, $\nabla f(x,y,z) = \langle ye^y + \sin z; , xe^y + xye^y, x\cos z\rangle$, and $f(1,1,0) = e$. Find $f(x,yz)$.

$f(x,yz) = xye^y + x\sin z$

(2) 3. Evaluate $\int_C ye^x\,ds$ where C is the top half of the unit circle.

$x = \cos t$

$y = \sin t$

$x' = \text{-}\sin t$

$y' = \cos t$

$$\int_C ye^x\,ds = \int_0^\pi (\sin t)e^{\cos t}\sqrt{\cos^2 t + \sin^2 t}\,dt = \int_0^\pi (\sin t)e^{\cos t}\,dt = \text{-}e^{\cos t}\Big|_0^\pi = e - \tfrac{1}{e}$$

(2) 4. Evaluate $\int_C xe^y\,ds$ where C is the part of the unit circle that lies in the first quadrant.

$x = \cos t$

$y = \sin t$

$x' = \text{-}\sin t$

$y' = \cos t$

$$\int_C xe^y\,ds = \int_0^{\frac{\pi}{2}} (\cos t)e^{\sin t}\sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{\frac{\pi}{2}} (\cos t)e^{\sin t}\,dt = e^{\sin t}\Big|_0^{\frac{\pi}{2}} = e - 1$$

(2) 5. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = \langle e^x \cos y, -e^x \sin y, 2z \rangle$ and C is the curve defined by
$\boldsymbol{r}(t) = \langle t^2, 2\pi t, 1 \rangle$ for $0 \leq t \leq 1$.

Set $f(x,y,z) = e^x \cos y + z^2$ and note that $\nabla f(x,y,z) = \boldsymbol{F}(x,y,z)$.

Then $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(1,2\pi,1) - f(0,0,1) = e + 1 - 2 = e - 1$.

(2) 6. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = \langle -e^y \sin x, e^y \cos x, 2z \rangle$ and C is the curve defined by
$\boldsymbol{r}(t) = \langle 2\pi t, t^2, 1 \rangle$ for $0 \leq t \leq 1$.

Set $f(x,y,z) = e^y \cos x + z^2$ and note that $\nabla f(x,y,z) = \boldsymbol{F}(x,y,z)$.

Then $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(2\pi,1,1) - f(0,0,1) = e + 1 - 2 = e - 1$.

(2) 7. Suppose that D is the region bounded by the circles centered at (0,0) with radii 1 and 2, C is the boundary of D, $P(x,y) = -x^2 y$, and $Q(x,y) = xy^2$. Evaluate $\int_C P\,dx + Q\,dy$.

$$\frac{\partial Q}{\partial x} = y^2$$

$$\frac{\partial P}{\partial y} = -x^2$$

$$\int_C P\,dx + Q\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^{2\pi} \int_1^2 r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \left. \tfrac{1}{4} r^4 \right|_1^2 d\theta$$

$$= \int_0^{2\pi} \tfrac{15}{4}\, d\theta$$

$$= \left. \tfrac{15}{4} \theta \right|_1^{2\pi}$$

$$= \tfrac{15\pi}{2}$$

(2) 8. Suppose that D is the region bounded by the circles centered at (0,0) with radii 1 and 3, C is the boundary of D, $P(x,y) = -x^2 y$, and $Q(x,y) = xy^2$. Evaluate $\int_C P\,dx + Q\,dy$.

$$\frac{\partial Q}{\partial x} = y^2$$

$$\frac{\partial P}{\partial y} = -x^2$$

$$\int_C P\,dx + Q\,dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^{2\pi} \int_1^3 r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^3 r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \left. \tfrac{1}{4} r^4 \right|_1^3 d\theta$$

$$= \int_0^{2\pi} 20\, d\theta$$

$$= \left. 20\,\theta \right|_0^{2\pi}$$

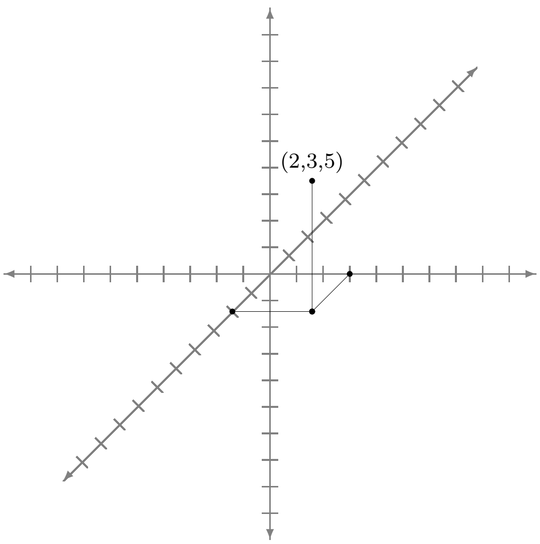
$$= 40\pi$$

Quiz 72 Name:________________________

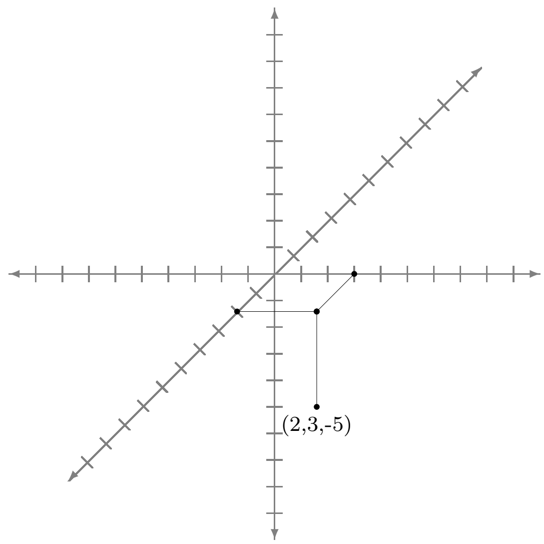
Directions: Show all of your work and justify all of your answers.

1. Plot each of the following points.

(1) a. (2,3,5)



(1) b. (2,3,-5)



2. Describe or draw the region defined by the given equations and inequalities.

(1) a. $x^2 + y^2 = 4$, $0 \leq z \leq 3$

Cylinder. The base is the circle centered at (0,0) of radius 2 in the xy-plane. The height of the cylinder is 3.

(1) b. $1 < x^2 + y^2 + z^2 < 4$

This is the region outside the sphere centered at the origin of radius 1 and inside the sphere centered at the origin of radius 2.

(2) 3. Give the equation of the sphere with center (1,-3,8) that has the point (2,-1,4) on its surface.

$r = d\,[(1,\text{-}3,8),(2,\text{-}1,4)] = \sqrt{(1-2)^2 + (\text{-}3 - \text{-}1)^2 + (8-4)^2} = \sqrt{21}$

$(x-1)^2 + (y+3)^2 + (z-8)^2 = 21$

4. Set $\boldsymbol{v} = \langle 3{,}2{,}\sqrt{3}\rangle$.

(1) a. Find a unit vector parallel to $\boldsymbol{v}$.

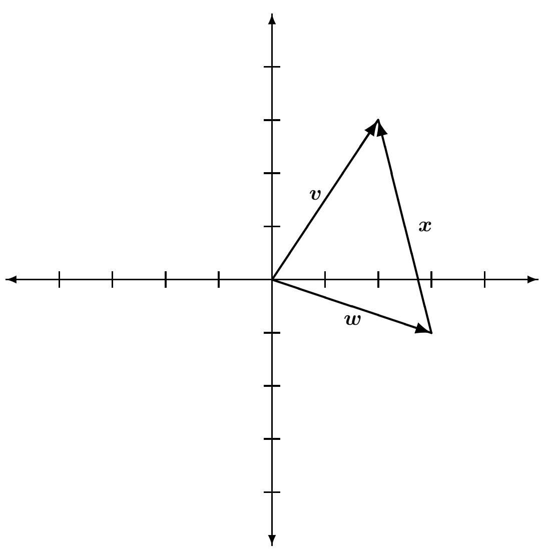
$\|\boldsymbol{v}\| = \sqrt{9+4+3} = 4$

$\frac{1}{4}\boldsymbol{v} = \frac{1}{4}\langle 3{,}2{,}\sqrt{3}\rangle$

(1) b. Find a vector with magnitude 5 that is parallel to $\boldsymbol{v}$.

$\frac{5}{4}\boldsymbol{v} = \frac{5}{4}\langle 3{,}2{,}\sqrt{3}\rangle$

(2) 5. Consider the figure below. Express $\boldsymbol{x}$ in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$. Give the components of $\boldsymbol{x}$.



$\boldsymbol{w} + \boldsymbol{x} = \boldsymbol{v}$

$\boldsymbol{x} = \boldsymbol{v} - \boldsymbol{w}$

$\boldsymbol{v} = \langle 2{,}3\rangle$

$\boldsymbol{w} = \langle 3{,}\text{-}1\rangle$

$\boldsymbol{x} = \langle 2{,}3\rangle - \langle 3{,}\text{-}1\rangle = \langle \text{-}1{,}4\rangle$

Quiz 73 Name:________________________

<u>Directions:</u> Show all of your work and justify all of your answers.

1. Find α such that the vectors $\langle 1,\text{-}2,\alpha\rangle$ and $\langle \text{-}2,4,10\rangle$ are

(1) a. parallel

$\text{-}2\langle 1,\text{-}2,\alpha\rangle = \langle \text{-}2,4,10\rangle$

$\text{-}2\alpha = 10$

$\alpha = \text{-}5$

(1) b. perpendicular

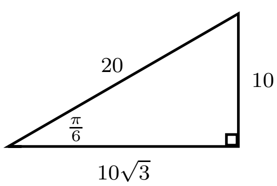
$\langle 1,\text{-}2,\alpha\rangle \cdot \langle \text{-}2,4,10\rangle = 0$

$\text{-}2 - 8 + 10\alpha = 0$

$\alpha = 1$

2. A wagon is pulled 50 ft by exerting a force of 20 lb on the handle at an angle of $\frac{\pi}{6}$ with the horizontal.

(1) a. Find the force vector.



$\boldsymbol{F} = \langle 10\sqrt{3},10\rangle$

(1) b. Find the displacement vector.

$\boldsymbol{D} = \langle 50,0\rangle$

(1) c. Calculate the work.

$\boldsymbol{F}\cdot\boldsymbol{D} = 500\sqrt{3}$

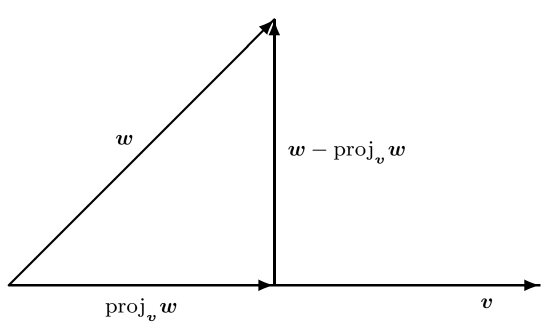
$500\sqrt{3}$ foot-pounds

3. Set $\boldsymbol{v} = \langle 1,\text{-}1,2\rangle$ and $\boldsymbol{w} = \langle \text{-}1,2,3\rangle$. Find each of the following.

(1) a. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{3}{\sqrt{6}}$

(1) b. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{3}{6}\boldsymbol{v} = \frac{1}{2}\langle 1,\text{-}1,2\rangle$

4. Consider the figure below.



(2) a. Draw and label the vectors $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$ and $\boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w}$.

(1) b. Find $\boldsymbol{v}\cdot\left(\boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w}\right)$. Hint: Use the picture you drew for the first part.

Since $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$ and $\boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w}$ are orthogonal, $\boldsymbol{v}\cdot\left(\boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w}\right) = 0$.

Quiz 74 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Set $\boldsymbol{v} = \langle 1,0,3 \rangle$ and $\boldsymbol{w} = \langle 2,\text{-}1,6 \rangle$. Calculate $\boldsymbol{v} \times \boldsymbol{w}$.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 3 \\ 2 & \text{-}1 & 6 \end{vmatrix} = \langle 3,0,\text{-}1 \rangle$$

2.
(3) a. Give the equation of the plane containing the points (-1,0,-4), (2,4,-7), and (-3,1,0).

Set $\boldsymbol{v} = \langle 2 - \text{-}1, 4-0, \text{-}7 - \text{-}4 \rangle = \langle 3,4,\text{-}3 \rangle$ and $\boldsymbol{w} = \langle \text{-}3 - \text{-}1, 1-0, 0 - \text{-}4 \rangle = \langle \text{-}2,1,4 \rangle$. Then a normal vector to the plane is $\boldsymbol{v} \times \boldsymbol{w}$.

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 4 & \text{-}3 \\ \text{-}2 & 1 & 4 \end{vmatrix} = \langle 19,\text{-}6,11 \rangle$$

$19(x+1) - 6y + 11(z+4) = 0$

(3) b. Give the equation of the line passing through the point (-1,1,0) in the direction of the vector $\boldsymbol{v} = \langle 1,2,\text{-}1 \rangle$.

Parametric: $x = \text{-}1 + t$ $y = 1 + 2t$ $z = \text{-}t$

(1) c. Find the point(s) if any where the plane and the line intersect.

$19(x+1) - 6y + 11(z+4) = 0$ $t = \frac{19}{2}$

$19(\text{-}1 + t + 1) - 6(1+2t) + 11(\text{-}t) = 0$ $\left(\frac{17}{2}, 20, \text{-}\frac{19}{2}\right)$

$\text{-}4t + 38 = 0$

(1) 3. Give the equation of the plane that contains the set of all points that are equidistant from the points (1,-2,1) and (1,0,3).

Solution 1: The midpoint of the line segment joining the two points is (1,-1,2). The vector $\langle 1-1, 0 - \text{-}2, 3-1 \rangle = \langle 0,2,2 \rangle = 2\langle 0,1,1 \rangle$ is orthogonal to the plane. The equation of the plane containing the point (1,-1,2) with normal vector $\langle 0,1,1 \rangle$ is $0(x-1) + (y+1) + (z-2) = 0$ which simplifies to $y + z = 1$.

Solution 2: If a point (x,y,z) is equidistant from (1,-2,1) and (1,0,3). then we have the following.

$$\sqrt{(x-1)^2 + (y+2)^2 + (z-1)^2} = \sqrt{(x-1)^2 + (y-0)^2 + (z-3)^2}$$

$$(x-1)^2 + (y+2)^2 + (z-1)^2 = (x-1)^2 + (y-0)^2 + (z-3)^2$$

$$y + z = 1$$

Quiz 75 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{r}(t) = \langle t^2+1, \cos \pi t, e^t \rangle$.

(3) a. Calculate $\boldsymbol{r}'(t)$.

$\boldsymbol{r}'(t) = \langle 2t, \text{-}\pi \sin \pi t, e^t \rangle$

(2) b. Give the equation of the line that is tangent to the graph of $\boldsymbol{r}$ at the point (1,1,1).

$\boldsymbol{r}(0) = \langle 1,1,1 \rangle$

$\boldsymbol{r}'(0) = \langle 0,0,1 \rangle$

$\boldsymbol{l}(t) = \langle 1,1,1 \rangle + t\langle 0,0,1 \rangle$

(3) c. Calculate $\int_0^1 \boldsymbol{r}(t)\,dt$.

$$\int_0^1 \boldsymbol{r}(t)\,dt = \int_0^1 \langle t^2+1, \cos \pi t, e^t \rangle\,dt = \left\langle \tfrac{1}{3}t^3 + t, \tfrac{1}{\pi}\sin \pi t, e^t \right\rangle \Big|_0^1 = \left\langle \tfrac{4}{3}, 0, e-1 \right\rangle$$

(1) 2. Calculate the following limit. Hint: Recall L'Hôpital's Rule.

$\lim_{t\to 0} \langle \frac{\sin t}{t}, \frac{1-e^t}{t}, \cos t \rangle = \langle 1, \text{-}1, 1 \rangle$

(1) 3. Give an equation for the tangent line to the curve of intersection of the cylinders $x^2+y^2=1$ and $x^2+z^2=1$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$.

First, find an equation for the curve. From the equations above, we have $y = \sqrt{1-x^2}$ and $z = \sqrt{1-x^2}$. So $\boldsymbol{r}(t) = \left\langle t, \sqrt{1-t^2}, \sqrt{1-t^2} \right\rangle$. Note that $\boldsymbol{r}\left(\frac{1}{2}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\rangle$. Alternatively, an equation of the curve is $\boldsymbol{r}(\theta) = \langle \cos\theta, \sin\theta, \sin\theta \rangle$. Note that $\boldsymbol{r}\left(\frac{\pi}{3}\right) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

$\boldsymbol{r}'(\theta) = \langle \text{-}\sin\theta, \cos\theta, \cos\theta \rangle$

$\boldsymbol{r}'\left(\frac{\pi}{3}\right) = \left\langle \text{-}\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$

$\boldsymbol{l}(t) = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\rangle + t\left\langle \text{-}\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$

Quiz 76 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. The motion of an object is described by $\boldsymbol{r}(t) = \left\langle 5\cos t, \sqrt{11}t, 5\sin t \right\rangle$.

(2) a. Find the unit tangent vector $\boldsymbol{T}(t)$.

$\boldsymbol{r}'(t) = \left\langle -5\sin t, \sqrt{11}, 5\cos t \right\rangle$.

$\|\boldsymbol{r}'(t)\| = 6$

$\boldsymbol{T}(t) = \frac{1}{6}\left\langle -5\sin t, \sqrt{11}, 5\cos t \right\rangle$

(1) b. Find the unit normal vector $\boldsymbol{N}(t)$.

$\boldsymbol{T}'(t) = \frac{1}{6}\langle -5\cos t, 0, -5\sin t\rangle$

$\|\boldsymbol{T}'(t)\| = \frac{5}{6}$

$\boldsymbol{N}(t) = \langle -\cos t, 0, -\sin t\rangle$

(1) c. Find the binormal vector $\boldsymbol{B}(t)$.

$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t) = \frac{1}{6}\left\langle -\sqrt{11}\sin t, -5, \sqrt{11}\cos t \right\rangle$

(1) d. Find the curvature $\kappa(t)$.

$\frac{\|\boldsymbol{T}'(t)\|}{\|\boldsymbol{r}'(t)\|} = \frac{5}{36}$

(1) e. Find the arc length of the curve from $t = 0$ to $t = 4$.

$$\int_0^4 \|\boldsymbol{r}'(t)\|\,dt = \int_0^4 6\,dt = 6t\Big|_0^4 = 24$$

(1) f. Reparameterize the curve with respect to arc length beginning at the point $t = 0$.

$$s = \int_0^t \|\boldsymbol{r}'(q)\|\,dq = \int_0^t 6\,dq = 6q\Big|_0^t = 6t$$

$t = \frac{1}{6}s$

$\boldsymbol{r}\left(\frac{s}{6}\right) = \left\langle 5\cos\left(\frac{s}{6}\right), \frac{\sqrt{11}s}{6}, 5\sin\left(\frac{s}{6}\right) \right\rangle$

(1) g. Find the velocity of the object when $t = \frac{\pi}{6}$.

$\boldsymbol{v}(t) = \boldsymbol{r}'(t) = \left\langle -5\sin t, \sqrt{11}, 5\cos t \right\rangle$

$\boldsymbol{v}\left(\frac{\pi}{6}\right) = \left\langle -\frac{5}{2}, \sqrt{11}, \frac{5\sqrt{3}}{2} \right\rangle$

(1) h. Find the acceleration of the object when $t = \frac{\pi}{6}$.

$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \langle -5\cos t, 0, -5\sin t\rangle$

$\boldsymbol{a}\left(\frac{\pi}{6}\right) = -5\left\langle \frac{\sqrt{3}}{2}, 0, \frac{1}{2} \right\rangle$

(1) i. Find the speed of the object when $t = \frac{\pi}{6}$.

$v(t) = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\| = 6$

$v\left(\frac{\pi}{6}\right) = 6$

Quiz 77 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. For the function below, give the domain and range and sketch the graph.

$f(x,y) = \sqrt{x^2 + y^2}$

D: $\mathbb{R}^3$

R: $[0,\infty)$

The graph is the top half of the cone $z^2 = x^2 + y^2$.

2. Calculate the following limits.

(2) a. $\lim\limits_{(x,y)\to(1,3)} \dfrac{x^2y + x}{x^3 - 2} =$ -4

(3) b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y + x^2y^2}{x^4 + y^4}$

$$\lim_{(x,0)\to(0,0)} \frac{x^3y + x^2y^2}{x^4 + y^4} = \lim_{y\to 0} \frac{0}{y^4} = 0$$

$$\lim_{(x,x)\to(0,0)} \frac{x^3y + x^2y^2}{x^4 + y^4} = \lim_{x\to 0} \frac{2x^4}{2x^4} = 1$$

So the limit does not exist.

(2) c. $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy - y}{\sqrt{x} - 1} = \lim\limits_{(x,y)\to(1,1)} \dfrac{y(x-1)}{\sqrt{x} - 1} = \lim\limits_{(x,y)\to(1,1)} y(\sqrt{x} + 1) = 2$

(1) d. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{\sqrt{x^2 + y^2}} = 0$

Proof: To see that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{\sqrt{x^2 + y^2}} = 0$, let $\varepsilon > 0$ be given. Set $\delta = \varepsilon$. Now suppose that $d[(x,y),(0,0,)] < \delta$.

Then

$$\sqrt{(x-0)^2 + (y-0)^2} < \delta$$

$$\sqrt{x^2 + y^2} < \delta$$

$$\sqrt{x^2} \leq \sqrt{x^2 + y^2} < \delta$$

$$|x| \leq \sqrt{x^2 + y^2} < \delta$$

$$|x| \cdot |x| \leq |x|\sqrt{x^2 + y^2}$$

$$x^2 < \delta\sqrt{x^2 + y^2} \qquad \text{(Since } |x| < \delta.\text{)}$$

$$\frac{x^2}{\sqrt{x^2 + y^2}} < \delta$$

$$\left| \frac{x^2}{\sqrt{x^2 + y^2}} - 0 \right| < \delta = \varepsilon$$

as desired. ∎

Quiz 78 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $f(x,y) = x^2y - xy^2$.

(6) a. Compute each of the following partial derivatives.

i. $f_x(x,y) = 2xy - y^2 = y(2x - y)$

ii. $f_y(x,y) = x^2 - 2xy = x(x - 2y)$

iii. $f_{xx}(x,y) = 2y$

iv. $f_{yy}(x,y) = \text{-}2x$

v. $f_{xy}(x,y) = 2x - 2y$

vi. $f_{yx}(x,y) = 2x - 2y$

(1) b. Find the critical points of f.

The only critical point is (0,0).

If $f_x(x,y) = y(2x - y) = 0$, then $y = 0$ or $y = 2x$. The only solution to $f_y(x,0) = 0$ is 0. So (0,0) is a critical point. Also, the only solution to $f_y(x,2x) = 0$ is 0. So (0,0) is the only critical point.

(1) c. Apply the Second Partials Test to the Critical Points. Find all local extrema. List the critical points for which the Second Partials Test is inconclusive.

Since $f_{xx}(0,0) = f_{yy}(0,0) = f_{xy}(0,0) = 0$, $D(0,0) = 0$. So the Second Partials Test is inconclusive.

(2) d. Let T be the region bounded by the triangle with vertices (0,0), (1,0), and (1,1). Find the maximum and minimum values of f on T.

Recall that the only critical point of f is (0,0) which is also on the boundary of T. So the extreme values of f on T will occur on the boundary of T.

Case 7: $y = 0$ and $0 \leq x \leq 1$

$f(x,0) = 0$

Case 8: $x = 1$ and $0 \leq y \leq 1$

Note $f(1,y) = y - y^2 = y(1 - y)$ which is a parabola with vertex $\left(\frac{1}{2},\frac{1}{4}\right)$. So the maximum value of f on the line segment is $f\left(1,\frac{1}{2}\right) = \frac{1}{4}$ and the minimum value of f on the line segment is $f(1,0) = f(1,1) = 0$.

Case 9: $y = x$ and $0 \leq x \leq 1$

$f(x,x) = 0$

So the minimum value of f on T is 0 and occurs at every point on the boundary where $y = 0$ or $y = x$ and the minimum value of f on T is $\frac{1}{4}$ and occurs at the point $\left(1,\frac{1}{2}\right)$.

Quiz 79 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $f(x,y) = y\sin x + e^{xy} + y$.

(3) a. Give the equation of the plane that is tangent to the surface $z = f(x,y)$ at the point (0,1,2).

$f_x(x,y) = y\cos x + ye^{xy}$

$f_y(x,y) = \sin x + xe^{xy} + 1$

$f_x(0,1) = 2$

$f_y(0,1) = 1$

$z - 2 = 2(x-0) + 1(y-1)$

$z = 2x + y + 1$

(2) b. Give the linear approximation of f at the point (0,1) and use it to approximate $f(0.01,0.99)$.

$L(x,y) = 2x + y + 1$

$L(0.01,0.99) = 0.02 + .99 + 1 = 2.01$

2. Consider the function $z = f(x,y) = x^2y^3$ where $x = s^2 + 3t$ and $y = \sin s - t$. Calculate each of the following.

(1) a. $\frac{\partial z}{\partial s}$

$\frac{\partial z}{\partial x} = 2xy^3$

$\frac{\partial z}{\partial y} = 3x^2y^2$

$\frac{\partial x}{\partial s} = 2s$

$\frac{\partial x}{\partial t} = 3$

$\frac{\partial y}{\partial s} = \cos s$

$\frac{\partial y}{\partial t} = \text{-}1$

$\frac{\partial z}{\partial s}$

$= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$

$= (2xy^3)(2s) + (3x^2y^2)(\cos s)$

$= 8s(s^2+3t)(\sin s - t)^3 + 3(\cos s)(s^2+3t)^2(\sin s - t)^3$

(1) b. $\frac{\partial z}{\partial t}$

$\frac{\partial z}{\partial t}$

$= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$

$= (2xy^3)(3) + (3x^2y^2)(\text{-}1)$

$= 6(s^2+3t)(\sin s - t)^3 - 3(s^2+3t)^2(\sin s - t)^2$

3. Set $f(x,y) = x^2 - y + 1$.

(2) a. Show that f is differentiable at (a,b) for all $(a,b) \in \mathbb{R}^2$.

$f_x(x,y) = 2x$

$f_y(x,y) = \text{-}1$

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - \left[f_x(a,b)\cdot h + f_y(a,b)\cdot k\right]}{\|(h,k)\|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{(a+h)^2 - (b+k) + 1 - (a^2 - b + 1) - (2ah - k)}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{a^2 + 2ah + h^2 - b - k + 1 - a^2 + b - 1 - 2ah + k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2+k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{h^2}{\sqrt{h^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2}{|h|}$$

$$= \lim_{(h,k)\to(0,0)} |h|$$

$$= 0$$

(1) b. Give $f'_{(a,b)}(x,y)$.

$f'_{(a,b)}(x,y) = \nabla f(a,b) \cdot \langle x,y\rangle = \langle 2a,\text{-}1\rangle \cdot \langle x,y\rangle = 2ax - y$

Quiz 80 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $f(x,y,z) = e^{xy} + y^2 + xz$.

(3) a. Calculate the gradient of f at (1,-1,2), $\nabla f(1,-1,2)$.

$\nabla f(x,y,z) = \langle ye^{xy} + z, xe^{xy} + 2y, x\rangle$

$\nabla f(1,-1,2) = \left\langle 2 - \frac{1}{e}, -2 + \frac{1}{e}, 1\right\rangle$

(2) b. Give the directional derivative of f at the point(1,-1,2) in the direction of $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right\rangle$.

$\left\langle 2 - \frac{1}{e}, -2 + \frac{1}{e}, 1\right\rangle \cdot \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right\rangle = \frac{1}{\sqrt{2}}$

(2) c. Give the equation of the tangent plane to the level surface $f(x,y,z) = 3 + \frac{1}{e}$ at the point (1,-1,2).

$(2 - \frac{1}{e})(x-1) + (-2 + \frac{1}{e})(y+1) + 1(z-2) = 0$

$\left\langle 2 - \frac{1}{e}, -2 + \frac{1}{e}, 1\right\rangle$

(1) d. Give the equation of the normal line to the level surface $f(x,y,z) = 3 + \frac{1}{e}$ at the point (1,-1,2).

$\boldsymbol{l}(t) = \langle 1,-1,2\rangle + t\left\langle 2 - \frac{1}{e}, -2 + \frac{1}{e}, 1\right\rangle$

(1) 2. Find the points on the unit sphere that are nearest to and farthest from the point (1,1,1).

Set $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$, $g(x,y,z) = x^2 + y^2 + z^2$, and find the maximum and minimum values of f subject to the constraint $g(x,y,z) = 1$. Use Lagrange multipliers.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle 2(x-1), 2(y-1), 2(z-1)\rangle = \lambda\langle 2x, 2y, 2z\rangle$

$x - 1 = \lambda x$

$x(1-\lambda) = 1$

$x = \frac{1}{1-\lambda}$

$y - 1 = \lambda y$

$y(1-\lambda) = 1$

$y = \frac{1}{1-\lambda}$

$z - 1 = \lambda z$

$z(1-\lambda) = 1$

$z = \frac{1}{1-\lambda}$

$x = y = z$

$x^2 + y^2 + z^2 = 1$

$3x^2 = 1$

$x = \pm\sqrt{\frac{1}{3}}$

$\left(\frac{1}{1-\lambda}\right)^2 = \frac{1}{3}$

$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = (1 - \sqrt{3})^2$

$f\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = (1 + \sqrt{3})^2$

(2) 3. Calculate $\displaystyle\iint_R x\sin(xy)\,dA$ where $R = \left[0,\frac{\pi}{2}\right] \times [0,1]$.

$$\iint_R x\sin(xy)\,dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 x\sin(xy)\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}} \text{-}\cos(xy)\Big|_0^1\, dx$$

$$= \int_0^{\frac{\pi}{2}} \big(\text{-}\cos x - \text{-}1\big)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \big(1 - \cos x\big)\,dx$$

$$= \big(x - \sin x\big)\Big|_0^{\frac{\pi}{2}}$$

$$= \tfrac{\pi}{2} - 1$$

Quiz 81 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Evaluate the following iterated integral.

$$\int_0^{\pi}\int_0^{\sin x}(\cos x)e^y\,dy\,dx = \int_0^{\pi}(\cos x)e^y\Big|_0^{\sin x}\,dx = \int_0^{\pi}\big[(\cos x)e^{\sin x} - \cos x\big]dx = \big(e^{\sin x} - \sin x\big)\Big|_0^{\pi}$$
$$= 0$$

(4) 2. The region R is bounded by the curves $x = 0$, $y = 1$, and $y = \sqrt{x}$. Give $\displaystyle\iint_R \frac{1}{y^3+1}\,dA$ as iterated integrals for both orders of integration. Evaluate one of the iterated integrals.

$$\int_0^1\int_{\sqrt{x}}^1 \frac{1}{y^3+1}\,dy\,dx$$

$$\int_0^1\int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy$$

$$\int_0^1\int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy = \int_0^1 \frac{x}{y^3+1}\Big|_0^{y^2}\,dy = \int_0^1 \frac{y^2}{y^3+1}\,dy = \ln|y^3+1|\Big|_0^1 = \ln 2$$

(3) 3. Evaluate the following iterated integral by converting to polar coordinates.

$$\int_0^1\int_0^{\sqrt{1-x^2}} xy^2\,dy\,dx$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 (r\cos\theta)(r^2\sin^2\theta)r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{5}r^5\cos\theta\sin^2\theta\right)\Big|_0^1\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \tfrac{1}{5}\cos\theta\sin^2\theta\,d\theta$$

$$= \tfrac{1}{15}\sin^3\theta\Big|_0^{\frac{\pi}{2}}$$

$$= \tfrac{1}{15}$$

Quiz 82 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Set T to be the solid tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$. Calculate the following integral.

$$\iiint\limits_T e^z \, dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} e^z \Big|_0^{1-x-y} \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left(e^{1-x-y} - 1\right) dy \, dx$$

$$= \int_0^1 \left(\text{-}e^{1-x-y} - y\right)\Big|_0^{1-x} \, dx$$

$$= \int_0^1 \left[\text{-}1 - (1-x) - (\text{-}e^{1-x} - 0)\right] dx$$

$$= \int_0^1 \left(e^{1-x} + x - 2\right) dx$$

$$= \left(\text{-}e^{1-x} + \tfrac{1}{2}x^2 - 2x\right)\Big|_0^1$$

$$= \text{-}1 + \tfrac{1}{2} - 2 - (\text{-}e + 0 - 0)$$

$$= e - \tfrac{5}{2}$$

2. Set R to be the region bounded by the half cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$.

(2) a. Give the integral $\iiint\limits_R z^2 \, dV$ as an iterated integral using rectangular coordinates. Do not evaluate the integral.

$$\iiint\limits_R z^2 \, dV = \int_{\text{-}1}^1 \int_{\text{-}\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 z^2 \, dz \, dy \, dx$$

(3) b. Give the integral $\iiint\limits_R z^2\,dV$ as an iterated integral using cylindrical coordinates and evaluate the integral.

$$\iiint\limits_R z^2\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_r^1 rz^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \tfrac{1}{3}rz^3\Big|_r^1\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \tfrac{1}{3}r(1-r^3)\,dr\,d\theta$$

$$= \tfrac{1}{3}\int_0^{2\pi}\int_0^1 \left(r-r^4\right)dr\,d\theta$$

$$= \tfrac{1}{3}\int_0^{2\pi}\left(\tfrac{1}{2}r^2-\tfrac{1}{5}r^5\right)\Big|_0^1\,d\theta$$

$$= \tfrac{1}{3}\int_0^{2\pi}\tfrac{3}{10}\,d\theta$$

$$= \tfrac{1}{10}\theta\Big|_0^{2\pi}$$

$$= \tfrac{\pi}{5}$$

(3) c. Give the integral $\iiint\limits_R z^2\,dV$ as an iterated integral using spherical coordinates and evaluate the integral.

$$\iiint\limits_R z^2\,dV$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sec\phi}(\rho^2\cos^2\phi)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\tfrac{1}{5}\rho^5\sin\phi\cos^2\phi\Big|_0^{\sec\phi}\,d\theta\,d\phi$$

$$= \tfrac{1}{5}\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\sec^3\phi\sin\phi\,d\theta\,d\phi$$

$$= \tfrac{1}{5}\int_0^{\frac{\pi}{4}}\int_0^{2\pi}(\cos\phi)^{-3}\sin\phi\,d\theta\,d\phi$$

$$= \tfrac{1}{5}\int_0^{\frac{\pi}{4}}\theta\Big|_0^{2\pi}(\cos\phi)^{-3}\,\phi\sin\phi\,d\phi$$

$$= \tfrac{2\pi}{5}\int_0^{\frac{\pi}{4}}(\cos\phi)^{-3}\sin\phi\,d\phi$$

$$= \tfrac{2\pi}{5}\left[\tfrac{1}{2}(\cos\phi)^{-2}\right]\Big|_0^{\frac{\pi}{4}}$$

$$= \tfrac{\pi}{5}$$

Quiz 83 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$, $\nabla f(x,y,z) = \langle ye^x + xye^x, xe^x, y\cos z\rangle$, and $f(1,1,0) = e$. Find $f(x,yz)$.

$f(x,yz) = xye^x + y\sin z$

(2) 2. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$, $\nabla f(x,y,z) = \langle ye^y + \sin z; ,xe^y + xye^y, x\cos z\rangle$, and $f(1,1,0) = e$. Find $f(x,yz)$.

$f(x,yz) = xye^y + x\sin z$

(2) 3. Evaluate $\int_C ye^x\,ds$ where C is the top half of the unit circle.

$x = \cos t$

$y = \sin t$

$x' = \text{-}\sin t$

$y' = \cos t$

$$\int_C ye^x\,ds = \int_0^{\pi} (\sin t)e^{\cos t}\,dt$$

$$= \int_0^{\pi} (\sin t)e^{\cos t}\sqrt{\cos^2 t + \sin^2 t}\,dt = \text{-}e^{\cos t}\Big|_0^{\pi}$$

$$= e - \tfrac{1}{e}$$

(2) 4. Evaluate $\int_C xe^y\,ds$ where C is the part of the unit circle that lies in the first quadrant.

$x = \cos t$

$y = \sin t$

$x' = \text{-}\sin t$

$y' = \cos t$

$$\int_C xe^y\,ds = \int_0^{\frac{\pi}{2}} (\cos t)e^{\sin t}\,dt$$

$$= \int_0^{\frac{\pi}{2}} (\cos t)e^{\sin t}\sqrt{\sin^2 t + \cos^2 t}\,dt = e^{\sin t}\Big|_0^{\frac{\pi}{2}}$$

$$= e - 1$$

(2) 5. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = \langle e^x \cos y, \text{-}e^x \sin y, 2z \rangle$ and C is the curve defined by
$\boldsymbol{r}(t) = \langle t^2, 2\pi t, 1 \rangle$ for $0 \leq t \leq 1$.

Set $f(x,y,z) = e^x \cos y + z^2$ and note that $\nabla f(x,y,z) = \boldsymbol{F}(x,y,z)$.

Then $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(1,2\pi,1) - f(0,0,1) = e + 1 - 2 = e - 1$.

(2) 6. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ where $\boldsymbol{F}(x,y,z) = \langle \text{-}e^y \sin x, e^y \cos x, 2z \rangle$ and C is the curve defined by
$\boldsymbol{r}(t) = \langle 2\pi t, t^2, 1 \rangle$ for $0 \leq t \leq 1$.

Set $f(x,y,z) = e^y \cos x + z^2$ and note that $\nabla f(x,y,z) = \boldsymbol{F}(x,y,z)$.

Then $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(2\pi,1,1) - f(0,0,1) = e + 1 - 2 = e - 1$.

(2) 7. Suppose that D is the region bounded by the circles centered at (0,0) with radii 1 and 2, C is the boundary of D, $P(x,y) = \text{-}x^2 y$, and $Q(x,y) = xy^2$. Evaluate $\int_C P\,dx + Q dy$.

$$\frac{\partial Q}{\partial x} = y^2$$

$$\frac{\partial P}{\partial y} = \text{-}x^2$$

$$\int_C P\,dx + Q dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^{2\pi} \int_1^2 r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4} r^4 \Big|_1^2\, d\theta$$

$$= \int_0^{2\pi} \tfrac{15}{4}\, d\theta$$

$$= \tfrac{15}{4} \theta \Big|_1^{2\pi}$$

$$= \tfrac{15\pi}{2}$$

(2) 8. Suppose that D is the region bounded by the circles centered at (0,0) with radii 1 and 3, C is the boundary of D, $P(x,y) = \text{-}x^2 y$, and $Q(x,y) = xy^2$. Evaluate $\int_C P\,dx + Q dy$.

$$\frac{\partial Q}{\partial x} = y^2$$

$$\frac{\partial P}{\partial y} = \text{-}x^2$$

$$\int_C P\,dx + Q dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (x^2 + y^2)\, dA$$

$$= \int_0^{2\pi} \int_1^3 r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_1^3 r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4} r^4 \Big|_1^3\, d\theta$$

$$= \int_0^{2\pi} 20\, d\theta$$

$$= 20\, \theta \Big|_0^{2\pi}$$

$$= 40\pi$$

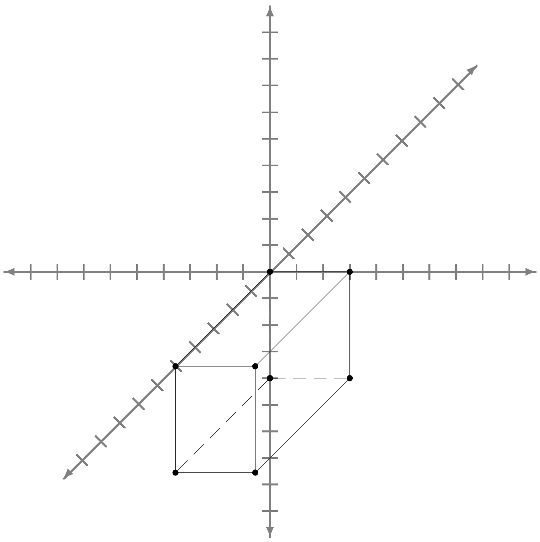
Chapter 7: Summer 2024

Section 7.1: Quizzes

Quiz 1 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. The top of the rectangular prism below lies in the xy-plane and its height is 4. Give the coordinates of the eight corners.



(0,0,0)

(5,0,0)

(0,3,0)

(5,3,0)

(0,0,-4)

(5,0,-4)

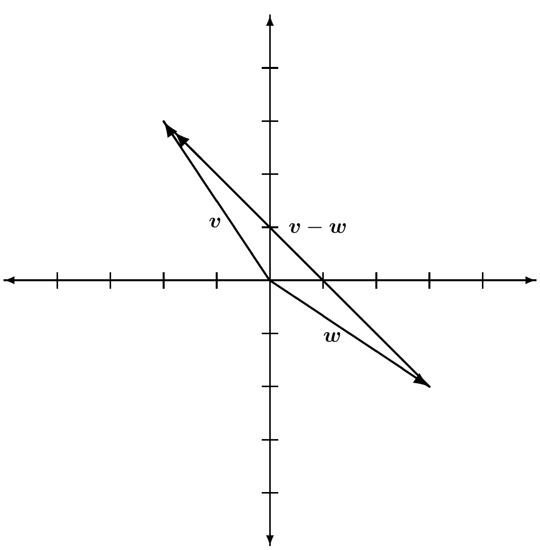
(0,3,-4)

(5,3,-4)

(1) 2. A sphere with radius 4 is sitting on the xy-plane at the point (-1,2,0). Give the equation of the sphere. Hint: Where is the center?

$(x + 1)^2 + (y - 2)^2 + (z - 4)^2 = 16$

(1) 3. In the figure below, draw the vector $\boldsymbol{v} - \boldsymbol{w}$.



Quiz 2 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Let $\boldsymbol{v} = \langle\text{-}1,3,4\rangle$, $\boldsymbol{w} = \langle 2,0,\text{-}1\rangle$, and θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Compute each of the following.

a. $\boldsymbol{v} \cdot \boldsymbol{w} = \text{-}6$ **b.** $\|\boldsymbol{v}\| = \sqrt{26}$ **c.** $\|\boldsymbol{w}\| = \sqrt{5}$

d. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \text{-}\frac{6}{\sqrt{130}}$ **e.** $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \text{-}\frac{6}{\sqrt{26}}$ **f.** $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v} = \text{-}\frac{3}{13}\langle\text{-}1,3,4\rangle$

(1) 2. Find the work done by a force of $\boldsymbol{F} = \langle 2,3,\text{-}5\rangle$ that moves an object from the point (-1,4,7) to the point (4,5,2). Force is measured in pounds and distance in feet.

Set $\boldsymbol{D} = \langle 5,1,\text{-}5\rangle$. Then the work done is $\boldsymbol{F}\cdot\boldsymbol{D}$ ft-lb = 38 ft-lb.

Quiz 3 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Find a vector that is orthogonal to both $\boldsymbol{v} = \langle -1,4,2 \rangle$ and $\boldsymbol{w} = \langle 2,1,3 \rangle$.

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 4 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 10\boldsymbol{i} + 7\boldsymbol{j} - 9\boldsymbol{k}$$

(1) 2. Find the volume of the parallelepiped determined by the vectors $\langle 1,0,2 \rangle$, $\langle 1,0,0 \rangle$, and $\langle 1,-1,1 \rangle$.

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 2$$

(1) 3. A .75 ft wrench is attached to a bolt and is parallel to the ground. A force of 30 lb is applied straight down. Give the magnitude of the torque about the center of the bolt.

$$\left(\tfrac{3}{4} \cdot 30 \cdot \sin \tfrac{\pi}{2}\right) \text{ lb} = 22.5 \text{ lb}$$

Quiz 4 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Consider the following.

P_1 : The plane with normal vector $\boldsymbol{n}_1 = \langle$-1,2,-4$\rangle$ containing the point (-2, 1, 5).

P_2 : The plane with normal vector $\boldsymbol{n}_2 = \langle$2,-1,3$\rangle$ containing the point (-2, 1, 5).

l_1 : The line parallel to the vector $\boldsymbol{v} = \langle$-3,1,4$\rangle$ containing the point (-2, 1, 5).

l_2 : The line of intersection of P_1 and P_2.

(1) a. Give the equation of P_1.

$-1(x+2) + 2(y-1) - 4(z-5) = 0$

$-x + 2y - 4z = -16$

(1) b. Give the equation of P_2.

$2(x+2) - (y-1) + 3(z-5) = 0$

$2x - y + 3z = 10$

(1) c. Give the equation(s) of l_1 in the form of your choice.

Parametric:

$x = -2 - 3t$

$y = 1 + t$

$z = 5 + 4t$

Symmetric:

$$-\frac{x+2}{3} = y - 1 = \frac{z-5}{4}$$

Vector:

$\boldsymbol{r}(t) = t\langle -3,1,4\rangle + \langle -2,1,5\rangle$

(2) d. Give the equation(s) of l_2 in the form of your choice. Hint: One point on the line is (-2,1,5). Set $x = 0$ to find another.

Set $x = 0$.

$2y - 4z = -16$

$-y + 3z = 10$

$y - 2z = -8$

$-y + 3z = 10$

$2x - y = 10$

$z = 2$

$y = -4$

So the line contains the points (-2,1,5) and (0,-4,2).

Set $\boldsymbol{v} = \langle -2,5,3\rangle$.

Parametric:

$x = -2 - 2t$

$y = 1 + 5t$

$z = 5 + 3t$

Symmetric:

$$\frac{x+2}{-2} = \frac{y-1}{5} = \frac{z-5}{3}$$

Vector:

$\boldsymbol{r}(t) = t\langle -2,5,3\rangle + \langle -2,1,5\rangle$

Quiz 5 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Consider the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

a. Describe the curve in $\mathbb{R}^2$ defined by the equation.

Ellipse.

b. Describe the surface in $\mathbb{R}^3$ defined by the equation.

Elliptic cylinder.

(3) 2. Match the equation with the surface.

a. $\frac{x^2}{4} - \frac{y^2}{9} + \frac{z^2}{25} = 1$

Hyperboloid of One Sheet

b. $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{25} = 1$

Hyperboloid of Two Sheets

c. $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$

Ellipsoid

d. $\frac{x^2}{4} + \frac{z^2}{9} = \frac{y}{25}$

Cone

e. $\frac{x^2}{4} - \frac{z^2}{9} = \frac{y}{25}$

Hyperbolic Paraboloid

f. $\frac{x^2}{4} + \frac{z^2}{9} = \frac{y^2}{25}$

Elliptic Paraboloid

(1) 3. Sketch each of the following vector curves in $\mathbb{R}^3$.

a. $\boldsymbol{f}(t) = \langle \cos t, \sin t, 0\rangle$

Unit circle.

b. $\boldsymbol{g}(t) = \langle \cos t, \sin t, t\rangle$

See class notes.

(1) 4. Do the following curves intersect? If so, where?

$\boldsymbol{f}(t) = \langle t^2, 3t-1, \sqrt{t}\rangle$

$\boldsymbol{g}(t) = \langle t, 2t^2, 2t\rangle$

No.

$t^2 = t$

$t = 0, 1$

$\boldsymbol{f}(0) = \langle 0,-1,0\rangle$

$\boldsymbol{g}(0) = \langle 0,0,0\rangle$

$\boldsymbol{f}(1) = \langle 1,2,1\rangle$

$\boldsymbol{g}(1) = \langle 1,2,2\rangle$

Quiz 6 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Calculate the following limit.

$\lim_{t\to 0}\langle e^t, \cos t, \frac{\sin t}{t}\rangle = \langle 1,1,1\rangle$

2. Differentiate.

(1) a. $\boldsymbol{f}(t) = \langle t^3, \sin t, e^t\rangle$

$\boldsymbol{f}'(t) = \langle 3t^2, \cos t, e^t\rangle$

(1) b. $\boldsymbol{g}(t) = \langle \ln t, e^{\sin t}, \sin^{-1} t\rangle$

$\boldsymbol{g}'(t) = \left\langle \frac{1}{t}, \cos t \cdot e^{\sin t}, \frac{1}{\sqrt{1-t^2}}\right\rangle$

(1) 3. Let $\boldsymbol{r}(t) = \left\langle t^3, \sqrt{t}, \cos \pi t\right\rangle$. Give an equation of the tangent line (in any form) to the curve at the point (1,1,-1).

$\boldsymbol{r}'(t) = \left\langle 3t^2, \frac{1}{2\sqrt{t}}, -\pi \sin \pi t\right\rangle$

$\boldsymbol{r}'(1) = \langle 3, \frac{1}{2}, 0\rangle$

$\boldsymbol{l}(t) = \langle 1,1,-1\rangle + t\langle 3, \frac{1}{2}, 0\rangle$

(1) 4. Integrate.

$\int_0^1 \left\langle e^t, \cos t, \frac{1}{1+x^2}\right\rangle dt = \langle e^t, \sin t, \tan^{-1} t\rangle\Big|_0^1 = \left\langle e-1, \sin 1, \frac{\pi}{4}\right\rangle$

Quiz 7 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{r}(t) = \langle \cos t, \sin t, 4t\rangle$ and let P be the point $(0, 1, 2\pi)$.

(1) a. Find the unit tangent vector at P.

$\boldsymbol{r}'(t) = \langle \text{-} \sin t, \cos t, 4\rangle$

$\|\boldsymbol{r}'(t)\| = \sqrt{17}$

$\boldsymbol{T}(t) = \frac{1}{\sqrt{17}} \langle \text{-} \sin t, \cos t, 4\rangle$

$\boldsymbol{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{17}} \langle \text{-}1, 0, 4\rangle$

(1) b. Find the unit normal vector at P.

$\boldsymbol{T}'(t) = \frac{1}{\sqrt{17}} \langle \text{-} \cos t, \text{-} \sin t, 0\rangle$

$\|\boldsymbol{T}'(t)\| = \frac{1}{\sqrt{17}}$

$\boldsymbol{N}(t) = \langle \text{-} \cos t, \text{-} \sin t, 0\rangle$

$\boldsymbol{N}\left(\frac{\pi}{2}\right) = \langle 0, \text{-}1, 0\rangle$

(1) c. Find the binormal vector at P.

$\boldsymbol{B}\left(\frac{\pi}{2}\right)$

$= \boldsymbol{T}\left(\frac{\pi}{2}\right) \times \boldsymbol{N}\left(\frac{\pi}{2}\right)$

$= \frac{1}{\sqrt{17}} \langle \text{-}1, 0, 4\rangle \times \langle 0, \text{-}1, 0\rangle$

$= \frac{1}{\sqrt{17}} \langle 4, 0, 1\rangle$

(1) d. Find the curvature at P.

$$\frac{\|\boldsymbol{T}'\left(\frac{\pi}{2}\right)\|}{\|\boldsymbol{r}'\left(\frac{\pi}{2}\right)\|} = \frac{\frac{1}{\sqrt{17}}}{\sqrt{17}} = \frac{1}{17}$$

(1) e. Reparameterize the curve with respect to arc length beginning at the point $(1, 0, 0)$ (different point than above).

$$s = s(t) = \int_0^t \|\boldsymbol{r}'(u)\|\, du = \int_0^t \sqrt{17}\, du = \sqrt{17}u\Big|_0^t = \sqrt{17}t$$

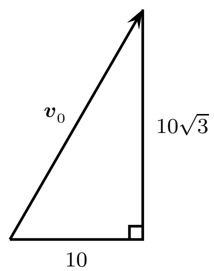
$t = \frac{s}{\sqrt{17}}$

$$\boldsymbol{r}\left(\frac{s}{\sqrt{17}}\right) = \left\langle \cos\left(\frac{s}{\sqrt{17}}\right), \sin\left(\frac{s}{\sqrt{17}}\right), \frac{4s}{\sqrt{17}}\right\rangle$$

Quiz 8 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. A projectile is launched due east with an initial speed of 20 feet per second at an angle of $\frac{\pi}{3}$ to the horizontal. Further, suppose that the acceleration due to gravity and wind blowing due north is given by $\boldsymbol{a}(t) = \langle\text{-}20,0,\text{-}32\rangle$. Give the vector function that represents the projectile's position at time t.



$\boldsymbol{v}_0 = \left\langle 0,10,10\sqrt{3}\right\rangle$

$\boldsymbol{a}(t) = \langle\text{-}20,0,\text{-}32\rangle$

$\boldsymbol{v}(t)$

$= \langle\text{-}20t,0,\text{-}32t\rangle + \boldsymbol{v}_0$

$= \left\langle\text{-}20t,10,\text{-}32t + 10\sqrt{3}\right\rangle$

$\boldsymbol{s}(t)$

$= \left\langle\text{-}10t^2,10t,\text{-}16t^2 + 10\sqrt{3}t\right\rangle + \boldsymbol{s}(0)$

$= \left\langle\text{-}10t^2,10t,\text{-}16t^2 + 10\sqrt{3}t\right\rangle + \mathbf{0}$

$= \left\langle\text{-}10t^2,10t,\text{-}16t^2 + 10\sqrt{3}t\right\rangle$

Quiz 9 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Give the domain and range. Sketch the graph.

a. $f(x,y) = \sqrt{x^2 + y^2}$

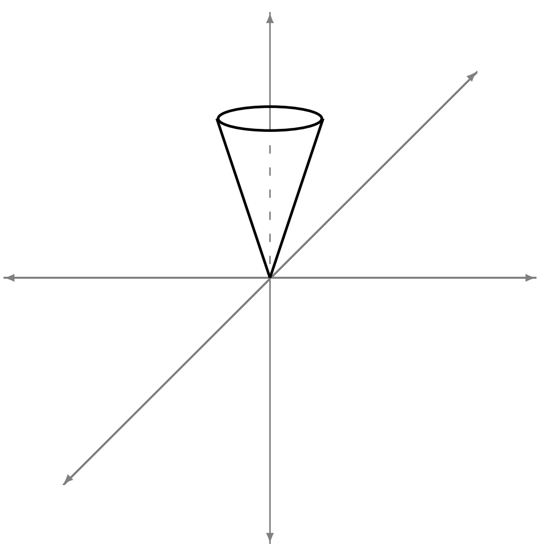
Domain: $\mathbb{R}^2$

Range: $[0,\infty)$

$z = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$

Top half of a cone.



b. $f(x,y) = \sqrt{x^2 + y^2 - 1}$

Domain: $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$ (i.e., the set of all points in the plane that are on or outside the unit circle)

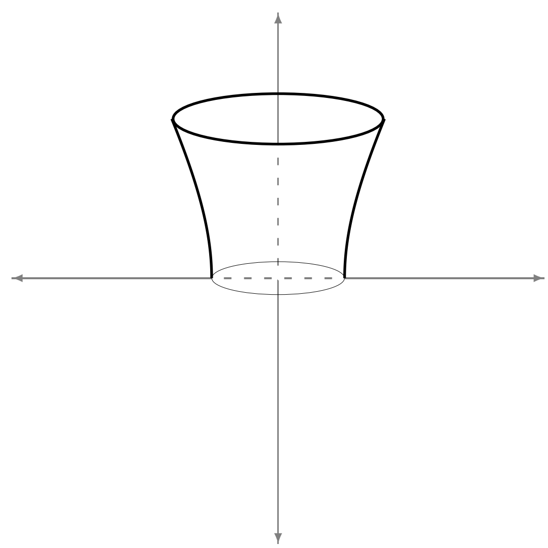
Range: $[0,\infty)$

$z = \sqrt{x^2 + y^2 - 1}$

$z^2 = x^2 + y^2 - 1$

$x^2 + y^2 - z^2 = 1$

Top half of a hyperboloid.



Quiz 10 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Calculate the following limits.

(1) a. $\lim_{(x,y)\to(0,0)} \frac{x^3y^2+y^2}{x^5+y}$

$$\lim_{(x,0)\to(0,0)} \frac{x^3y^2+y^2}{x^5+y} = \lim_{(x,0)\to(0,0)} \frac{0}{x^5} = 0$$

$$\lim_{(x,x)\to(0,0)} \frac{x^3y^2+y^2}{x^5+y} = \lim_{(x,x)\to(0,0)} \frac{x^5+x^2}{x^5+x} = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^2+y^2}{x^5+y} \text{ DNE}$$

(1) b. $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = 1$

Let $\theta = x + y$.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

(1) 2. Calculate all first order partial derivatives and all second order partial derivatives of f.

$f(x,y) = x^3y^2 + e^x + \sin y$

$f_x(x,y) = 3x^2y^2 + e^x$

$f_y(x,y) = 2x^3y + \cos y$

$f_{xx}(x,y) = 6xy^2 + e^x$

$f_{xy}(x,y) = 6x^2y$

$f_{yx}(x,y) = 6x^2y$

$f_{yy}(x,y) = 2x^3 - \sin y$

(1) 3. Calculate all first order partial derivatives of f.

$f(x,y,z) = xe^y\sin(xz)$

$f_x(x,y,z) = e^y\sin(xz) + xze^y\cos(xz)$

$f_y(x,y,z) = xe^y\sin(xz)$

$f_z(x,y,z) = x^2e^y\cos(xz)$

Quiz 11 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $f(x,y) = x^2y^3 + x^2 - y^3$.

(2) a. Find the critical points and apply the Second Partials Test to each one.

$f_x(x,y) = 2xy^3 + 2x = 2x(y^3 + 1)$

$f_y(x,y) = 3x^2y^2 - 3y^2 = 3y^2(x^2 - 1)$

$f_{xx}(x,y) = 2y^3 + 2$

$f_{xy}(x,y) = 6xy^2$

$f_{yx}(x,y) = 6xy^2$

$f_{yy}(x,y) = 6x^2y - 6y$

CP: (0,0), (1,-1), (-1,-1)

$D(0,0) = 0$

Inconclusive.

$D(1,\text{-}1) = 0 - 36 = \text{-}36$

Saddle Point: (1,-1,1)

$D(\text{-}1,\text{-}1) = 0 - 36 = \text{-}36$

Saddle Point: (-1,-1,1)

(1) b. Prove each of the following.

i. For each $y \in \mathbb{R}$, $f(1,y) = 1$.

***Proof*:** Note that $f(1,y) = y^3 + 1 - y^3 = 1$. ∎

ii. For each $x \in \mathbb{R}$, $f(x,\text{-}1) = 1$.

***Proof*:** Note that $f(x,\text{-}1) = \text{-}x^2 + x^2 + 1 = 1$. ∎

Quiz 12 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the linear approximation for the following function at the point (1,1,0).

$f(x,y,z) = x^3y^2 + xe^z$ $f(1,1,0) = 2$

$f_x(x,y,z) = 3x^2y^2 + e^z$ $f_x(1,1,0) = 4$

$f_y(x,y,z) = 2x^3y$ $f_y(1,1,0) = 2$

$f_z(x,y,z) = xe^z$ $f_z(1,1,0) = 1$

$L(x,y,z) = 2 + 4(x-1) + 2(y-1) + 1(z-0) = 4x + 2y + z - 4$

2. Recall the formulas for volume and surface area of a cylinder.

$V = \pi r^2 h$

$S = 2\pi r^2 + 2\pi rh$

Suppose that the radius of a cylinder is measured to be 2 inches and its height is measured to be 8 inches. Further suppose that the measurements are known to be accurate to within 0.01 inches.

(1) a. Use differentials to find the maximum error in calculating the volume of the cylinder using the given measurements.

$dV = (2\pi rh)dr + (\pi r^2)dh$

$= 32\pi \times .01 + 4\pi \times .01$

$= 0.36\pi$

(1) b. Use differentials to find the maximum error in calculating the surface area of the cylinder using the given measurements.

$dS = (4\pi r + 2\pi h)dr + (2\pi r)dh$

$= 24\pi \times .01 + 4\pi \times .01$

$= 0.28\pi$

Quiz 13 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Prove that $f(x,y) = 2x + y^2 + 1$ is differentiable for all $(a,b) \in \mathbb{R}^2$ and find $f'_{(a,b)}(x,y)$.

***Proof*:** Note that

$$f_x(x,y) = 2$$

and

$$f_y(x,y) = 2y.$$

Consider

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - \big(f_x(a,b)h + f_y(a,b)k\big)}{\|(h,k)\|}$$

$$= \lim_{(h,k)\to(0,0)} \frac{2(a+h) + (b+k)^2 + 1 - (2a + b^2 + 1) - (2h + 2bk)}{\sqrt{h^2 + k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{2a + 2h + b^2 + 2bk + k^2 + 1 - 2a - b^2 - 1 - 2h - 2bk}{\sqrt{h^2 + k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{h^2 + k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{k^2}{\sqrt{k^2}}$$

$$\leq \lim_{(h,k)\to(0,0)} \frac{k^2}{|k|}$$

$$= \lim_{(h,k)\to(0,0)} |k|$$

$$= 0$$

as desired. ■

$$f'_{(a,b)}(x,y) = 2x + 2by$$

Quiz 14 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $z = ye^x + x\ln y$ where $x = t^2 + 1$ and $y = \sqrt{t}$. Calculate $\frac{dz}{dt}$ at the point where $t = 1$.

$\frac{\partial z}{\partial x} = ye^x + \ln y$

$\frac{\partial z}{\partial y} = e^x + \frac{x}{y}$

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t}}$

$t = 1$

$x = 2$

$y = 1$

$\frac{dz}{dt} = (ye^x + \ln y)(2t) + \left(e^x + \frac{x}{y}\right)\left(\frac{1}{2\sqrt{t}}\right)$

$\frac{dz}{dt} = (e^2 + \ln 1)(2) + \left(e^2 + \frac{2}{1}\right)\left(\frac{1}{2\sqrt{1}}\right)$

$= \frac{5e^2}{2} + 1$

(2) 2. Let $z = 2x^3y^2$ where $x = e^s + \sin t$ and $y = (s+t)^2$. Calculate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the point where $(s,t) = (1,0)$.

$\frac{\partial z}{\partial x} = 6x^2y^2$

$\frac{\partial z}{\partial y} = 4x^3y$

$\frac{\partial x}{\partial s} = e^s$

$\frac{\partial x}{\partial t} = \cos t$

$\frac{\partial y}{\partial s} = 2(s+t)$

$\frac{\partial y}{\partial t} = 2(s+t)$

$(s,t) = (1,0)$

$x = e$

$y = 1$

$\frac{\partial z}{\partial s} = 6x^2y^2e^s + 8x^3y(s+t) = 14e^3$

$\frac{\partial z}{\partial t} = 6x^2y^2\cos t + 8x^3y(s+t) = 6e^2 + 8e^3$

Quiz 15 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Maximize the function $f(x,y,z) = xy + yz$ subject to the constraint $x + y + z = 3$. Show that the function has no minimum value subject to the constraint.

Set $g(x,y,z) = x + y + z - 3$.

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\nabla f(x,y,z) = = \langle y,x + z,y\rangle$

$\langle y,x + z,y\rangle = \lambda\langle 1,1,1\rangle$

$\nabla g(x,y,z) = \langle 1,1,1\rangle$

If $\lambda = 0$, then $y = 0$. Note that $f(x,0,z) = 0$. So $y = \lambda$ and $x + z = \lambda = y$.

$x + y + z = 3$

$y = \frac{3}{2}$

$2\lambda = 3$

$x + z = \frac{3}{2}$

$\lambda = \frac{3}{2}$

$f\left(x,\frac{3}{2},z\right) = \frac{3}{2}x + \frac{3}{2}z = \frac{3}{2}(x + z) = \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}$

***Proof*:** To see that the function has no minimum value subject to the constraint, let $x > 0$ and note that $x - x + 3 = 3$ and $f(x,\text{-}x,3) = \text{-}x^2 - 3x$. Therefore, $\lim_{x\to\infty} f(x,\text{-}x,3) = \text{-}\infty$. ■

(2) 2. Find the maximum and minimum values of the function $f(x,y,z) = xyz$ subject to the constraint $x^2 + 4y^2 + 9z^2 = 25$.

Set $g(x,y,z) = x^2 + 4y^2 + 9z^2 - 25$.

$\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$

$\nabla f(x,y,z) = \langle yz,xz,yz\rangle$

$\langle yz,xz,yz\rangle = \lambda\langle 2x,8y,18z\rangle$

$\nabla g(x,y,z) = \langle 2x,8y,18z\rangle$

If $\lambda = 0$, then two of x, y, and z are 0 which means that $f(x,y,z) = 0$.

$yz = 2\lambda x$ $\quad$ $xz = 8\lambda y$ $\quad$ $xy = 18\lambda z$

$xyz = 2\lambda x^2$ $\quad$ $xyz = 8\lambda y^2$ $\quad$ $xyz = 18\lambda z^2$

$3xyz = 2\lambda x^2 + 8\lambda y^2 + 18\lambda z^2 = 2\lambda(x^2 + 4y^2 + 9z^2) = 2\lambda(25) = 50\lambda$

$xyz = \frac{50}{3}\lambda$

$xyz = 2\lambda x^2$ $\quad$ $xyz = 8\lambda y^2$ $\quad$ $xyz = 18\lambda z^2$

$\frac{50}{3}\lambda = 2\lambda x^2$ $\quad$ $\frac{50}{3}\lambda = 8\lambda y^2$ $\quad$ $\frac{50}{3}\lambda = 18\lambda z^2$

$x = \pm\frac{5}{\sqrt{3}}$ $\quad$ $y = \pm\frac{5}{2\sqrt{3}}$ $\quad$ $z = \pm\frac{5}{3\sqrt{3}}$

Max: $= \frac{125}{18\sqrt{3}}$

Min: $= \text{-}\frac{125}{18\sqrt{3}}$

3. Consider the circle formed by intersecting the sphere $x^2 + y^2 + z^2 = 1$ and the plane $2x + 2y + 2z = 1$.

(2) a. Find the points on circle that are nearest to and farthest from the point $(1, 1, 2)$.

Set

$f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-2)^2$

$g(x,y,z) = 2x + 2y + 2z - 1$

$h(x,y,z) = x^2 + y^2 + z^2 - 1$

Then

$f(x,y,z)$

$= x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 - 4z + 4$

$= (x^2 + y^2 + z^2) - (2x + 2y + 2z) - 2z + 6$

$= 1 - 1 - 2z + 6$

$= 6 - 2z$

Consider

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

$\langle 0,0,\text{-}2\rangle = \lambda \langle 2,2,2\rangle + \mu \langle 2x,2y,2z\rangle$

$2\lambda + 2\mu x = 0$ $\quad$ $2\lambda + 2\mu y = 0$ $\quad$ $2\lambda + 2\mu z = \text{-}2$

$\lambda + \mu x = 0$ $\quad$ $\lambda + \mu y = 0$ $\quad$ $\lambda + \mu z = \text{-}1$

If $\mu = 0$, then $\lambda = 0$ since $2\lambda + 2\mu x = 0$. However, this is not possible since $\lambda + \mu z = \text{-}1$. Therefore, $\mu \neq 0$. Since $\lambda + \mu x = 0 = \lambda + \mu y$, $x = y$.

$2x + 2y + 2z = 1$

$4x + 2z = 1$

$z = \frac{1-4x}{2}$

$x^2 + y^2 + z^2 = 1$

$2x^2 + \left(\frac{1-4x}{2}\right)^2 = 1$

$8x^2 + (1 - 4x^2)^2 = 4$

$8x^2 + 16x^2 - 8x + 1 = 4$

$24x^2 - 8x - 3 = 0$

$x = \frac{2 \pm \sqrt{22}}{12}$

$f\left(\frac{2+\sqrt{22}}{12}, \frac{2+\sqrt{22}}{12}, \frac{1-\sqrt{22}}{6}\right) = \frac{17+\sqrt{22}}{3}$ (max)

$f\left(\frac{2-\sqrt{22}}{12}, \frac{2-\sqrt{22}}{12}, \frac{1+\sqrt{22}}{6}\right) = \frac{17-\sqrt{22}}{3}$ (min)

Nearest: $\left(\frac{2-\sqrt{22}}{12}, \frac{2-\sqrt{22}}{12}, \frac{1+\sqrt{22}}{6}\right)$

Farthest: $\left(\frac{2+\sqrt{22}}{12}, \frac{2+\sqrt{22}}{12}, \frac{1-\sqrt{22}}{6}\right)$

(2) b.

i. Prove that all points on the circle are equidistant from the point $(1, 1, 1)$.

Proof: Suppose that (r,s,t) is a point on the circle. Then

$(r-1)^2 + (s-1)^2 + (t-1)^2 = (r^2 + s^2 + t^2) - (2r + 2s + 2t) + 3 = 1 - 1 + 3 = 3$

which means that $d[(r,s,t),(1,1,1)] = \sqrt{3}$. ■

ii. Narrate an intuitive and/or geometric explanation.

A direction vector for the line containing the points (1,1,1) and (0,0,0) is $\langle 1,1,1 \rangle$ which is orthogonal to the plane $2x + 2y + 2z = 1$. So the line segments joining the point (1,1,1) to the points on the circle form a right circular cone and therefore all have the same length.

Quiz 16 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Set $R = [0,1] \times \left[0,\frac{\pi}{2}\right]$. Compute the following integral.

$$\iint\limits_R e^x \cos y \, dA = \int_0^1 \int_0^{\frac{\pi}{2}} e^x \cos y \, dy \, dx = \int_0^1 e^x \sin y \Big|_0^{\frac{\pi}{2}} \, dx = \int_0^1 e^x (1-0) \, dx = e^x \Big|_0^1 = e - 1$$

(1) 2. Let R be the region in $\mathbb{R}^2$ bounded by the curves $y = x^2 - 1$ and $y = \text{-}x^2 + 1$. Compute the following integral.

$$\iint\limits_R (x+4) \, dA$$

$$= \int_{-1}^{1} \int_{x^2-1}^{-x^2+1} (x+4) \, dy \, dx$$

$$= \int_{-1}^{1} (xy + 4y)\Big|_{x^2-1}^{-x^2+1} \, dx$$

$$= \int_{-1}^{1} \left(x(-x^2+1) + 4(-x^2+1) - [x(x^2-1) + 4(x^2-1)]\right) dx$$

$$= \int_{-1}^{1} \left(-2x^3 - 8x^2 + 2x + 8\right) dx$$

$$= \left(-\tfrac{1}{2}x^4 - \tfrac{8}{3}x^3 + x^2 + 8x\right)\Big|_{-1}^{1}$$

$$= -\tfrac{1}{2} - \tfrac{8}{3} + 1 + 8 - \left(-\tfrac{1}{2} + \tfrac{8}{3} + 1 - 8\right)$$

$$= \tfrac{32}{3}$$

(1) 3. Let R be the region in $\mathbb{R}^2$ bounded by the triangle with vertices (0,0), (1,1), and (0,1). Compute the following integral. Hint: The region is both type I and type II. Choose the order of integration carefully.

$$\iint\limits_R 6xe^{y^3} \, dA = \int_0^1 \int_0^y \left(6xe^{y^3}\right) dx \, dy = \int_0^1 3x^2 e^{y^3}\Big|_0^y \, dy = \int_0^1 3y^2 e^{y^3} \, dy = e^{y^3}\Big|_0^1 = e - 1$$

Quiz 17 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let D be the unit circle. Calculate the following integral.

$$\iint_D xy\,dA$$

$$= \int_0^{2\pi}\int_0^1 r(r\cos\theta)(r\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4}r^4\cos\theta\sin\theta\Big|_0^1\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{4}\cos\theta\sin\theta\,d\theta$$

$$= \tfrac{1}{8}\sin^2\theta\Big|_0^{2\pi}$$

$$= 0$$

(1) 2. Evaluate the following integral. Hint: Use polar coordinates.

$$\int_{-1}^1\int_0^{\sqrt{1-x^2}}\left(x+\sqrt{x^2+y^2}\right)dy\,dx$$

$$= \int_0^{\pi}\int_0^1 (r\cos\theta + r)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^1 r^2(\cos\theta+1)\,dr\,d\theta$$

$$= \int_0^{\pi} \tfrac{1}{3}r^3(\cos\theta+1)\Big|_0^1\,d\theta$$

$$= \tfrac{1}{3}\int_0^{\pi}\left(\cos\theta+1\right)d\theta$$

$$= \tfrac{1}{3}\left(\sin\theta+\theta\right)\Big|_0^{\pi}$$

$$= \tfrac{\pi}{3}$$

Quiz 18 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. For each of the following, give the coordinates of the point in the the other two coordinate systems.

a.
Rectangular: $\left(1,\text{-}1,\sqrt{2}\right)$

$r = \sqrt{2}$

$\theta = \frac{7\pi}{4}$

$\rho = 2$

$\phi = \frac{\pi}{4}$

Cylindrical: $\left(\sqrt{2},\frac{7\pi}{4},\sqrt{2}\right)$

Spherical: $\left(2,\frac{7\pi}{4},\frac{\pi}{4}\right)$

b.
Cylindrical: $\left(2,\frac{\pi}{3},\text{-}4\right)$

$x = 1$

$y = \sqrt{3}$

$\rho = 2\sqrt{5}$

$z = \rho\cos\phi$

$\text{-}4 = 2\sqrt{5}\cos\phi$

$\cos\phi = \text{-}\frac{2}{\sqrt{5}}$

$\phi = \cos^{\text{-}1}\left(\text{-}\frac{2}{\sqrt{5}}\right)$

Rectangular: $\left(1,\sqrt{3},\text{-}4\right)$

Spherical: $\left(2\sqrt{2},\frac{\pi}{3},\cos^{\text{-}1}\left[\text{-}\frac{2}{\sqrt{5}}\right]\right)$

c.
Spherical: $\left(3,\frac{3\pi}{4},\frac{\pi}{6}\right)$

$r = \frac{3}{2}$

$x = \text{-}\frac{3}{2\sqrt{2}}$

$y = \frac{3}{2\sqrt{2}}$

$z = \frac{3\sqrt{3}}{2}$

Rectangular: $\left(\text{-}\frac{3}{2\sqrt{2}},\frac{3}{2\sqrt{2}},\frac{3\sqrt{3}}{2}\right)$

Cylindrical: $\left(\frac{3}{2},\frac{3\pi}{4},\frac{3\sqrt{3}}{2}\right)$

(1) 2. Express the following equation using spherical coordinates. Simplify your answer completely. Identify the object.

$x^2 + y^2 = z^2$

$x^2 + y^2 + z^2 = 2z^2$

$\rho^2 = 2z^2$

$\rho^2 = 2\rho^2\cos^2\phi$

$\cos^2\phi = \frac{1}{2}$

$\cos\phi = \frac{1}{\sqrt{2}}$

$\phi = \frac{\pi}{4}$

Cone.

(1) 3. Compute the following integral where T is the solid tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), and (0,0,1).

The equztion of the plane containing the points (1,0,0), (0,1,0), and (0,0,1) is $x+y+z=1$.

$$\iiint_T y\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x} yz\Big|_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} y(1-x-y)\,dy\,dx = \int_0^1\int_0^{1-x}\left(y - xy - y^2\right)dy\,dx = \int_0^1 \left(\tfrac{1}{2}y^2 - \tfrac{1}{2}xy^2 - \tfrac{1}{3}y^3\right)\Big|_0^{1-x}\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}(1-x)^2 - \tfrac{1}{2}x(1-x)^2 - \tfrac{1}{3}(1-x)^3\right]dx = \int_0^1 \tfrac{1}{6}(1-x)^2\left[3-3x-2(1-x)\right]dx$$

$$= \int_0^1 \tfrac{1}{6}(1-x)^3\,dx = -\tfrac{1}{24}(1-x)^4\Big|_0^1 = \tfrac{1}{24}$$

(1) 4. Compute the following integral where $R = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 1, 0 \le z \le 1\right\}$.

$$\iiint_R x\,dV = \int_0^{2\pi}\int_0^1\int_0^1 r\cdot r\cos\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 zr^2\cos\theta\Big|_0^1\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{3}r^3\cos\theta\Big|_0^1\,d\theta = \int_0^{2\pi}\tfrac{1}{3}\cos\theta\,d\theta = \tfrac{1}{3}\sin\theta\Big|_0^{2\pi} = 0$$

(1) 5. Let $a > 0$. Compute the following integral. Hint: The integral can be computed without integrating.

$$\int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$z = \pm\sqrt{a^2-r^2}$$

$$z^2 = a^2 - r^2$$

$$z^2 + r^2 = a^2$$

$$x^2+y^2+z^2 = a^2$$

This is the sphere centered at (0,0,0) of radius a. The volume of the sphere is $\frac{4}{3}\pi a^3$.

Let S be the sphere above.

Then $\displaystyle\int_0^{2\pi}\int_0^a\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = \iiint_S 1\,dV$ which is the volume of the sphere.

Quiz 19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Evaluate the following integral. Hint: Use spherical coordinates.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} x\,dz\,dy\,dx$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho\sin\phi\cos\theta\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \tfrac{1}{4}\rho^4\sin^2\phi\cos\theta\Big|_0^1\,d\theta\,d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \tfrac{1}{4}\sin^2\phi\cos\theta\,d\theta\,d\phi$$

$$= \int_0^{\pi} \tfrac{1}{4}\sin^2\phi\sin\theta\Big|_0^{2\pi}\,d\phi$$

$$= \int_0^{\pi} 0\,d\phi$$

$$= 0$$

(1) 2. Let R be the part of the cone $z^2 = x^2 + y^2$ that lies above the region $x^2 + y^2 \leq 1$ in the xy-plane.

Evaluate the following integral. Hint: Use spherical coordinates.

$$\iiint_R \sqrt{x^2+y^2+z^2}\,dV$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{2}} \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \tfrac{1}{4}\rho^4\sin\phi\Big|_0^{\sqrt{2}}\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \sin\phi\,d\theta\,d\phi$$

$$= \int_0^{\frac{\pi}{4}} \theta\sin\phi\Big|_0^{2\pi}\,d\phi$$

$$= \int_0^{\frac{\pi}{4}} 2\pi\sin\phi\,d\phi$$

$$= \left(-2\pi\cos\phi\right)\Big|_0^{\frac{\pi}{4}}$$

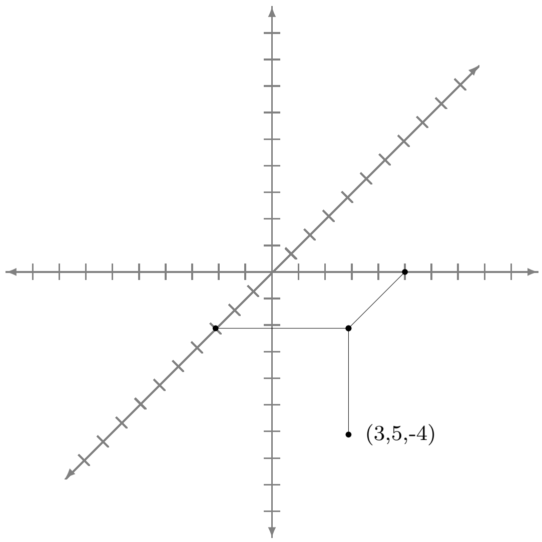
$$= -2\pi\left(\tfrac{1}{\sqrt{2}} - 1\right)$$

$$= 2\pi\left(1 - \tfrac{1}{\sqrt{2}}\right)$$

Section 7.2: Exam 1

(10) 1. Plot the following point.

(3,5,-4)



(20) 2. Give the equation of the sphere with center (1,-2,3) and radius 4.

$(x-1)^2 + (y+2)^2 + (z-3)^2 = 16$

(10) 3. Give the equation for the set of all points that are equidistant from the points (-1,1,2) and (3,2,-1). Describe the region defined by the set.

$(x+1)^2 + (y-1)^2 + (z-2)^2 = (x-3)^2 + (y-2)^2 + (z+1)^2$

$x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 - 4z + 4 = x^2 - 6x + 9 + y^2 - 4y + 4 + z^2 + 2z + 1$

$8x + 2y - 6z = 8$

$4x + y - 3z = 4$

Plane.

(120) 4. Set $\boldsymbol{v} = \langle \text{-}2,\text{-}1,4 \rangle$, $\boldsymbol{w} = \langle 1,\text{-}1,3 \rangle$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Compute each of the following.

a. $\boldsymbol{v} + \boldsymbol{w}$

$\langle \text{-}1,\text{-}2,7 \rangle$

b. $\boldsymbol{v} \cdot \boldsymbol{w}$

11

c. $\boldsymbol{v} \times \boldsymbol{w}$

$\langle 1,10,3 \rangle$

d. $\|\boldsymbol{v}\|$

$\sqrt{21}$

e. $\|\boldsymbol{w}\|$

$\sqrt{11}$

f. $\cos\theta$

$\frac{11}{\sqrt{21}\sqrt{11}} = \frac{\sqrt{11}}{\sqrt{21}}$

g. $\sin\theta$

$\frac{\sqrt{110}}{\sqrt{21}\sqrt{11}} = \frac{\sqrt{10}}{\sqrt{21}}$

h. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w}$

$\frac{11}{\sqrt{21}}$

i. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$

$\frac{11}{21}\langle \text{-}2,\text{-}1,4 \rangle$

j. The volume of the parallelepiped determined by the vectors $\boldsymbol{v}$, $\boldsymbol{w}$, and $\langle \text{-}1,0,1 \rangle$.

$$\begin{vmatrix} -2 & -1 & 4 \\ 1 & -1 & 3 \\ -1 & 0 & 1 \end{vmatrix} = \text{-}2(\text{-}1) - (\text{-}1)(4) + 4(\text{-}1) = 2$$

(10) 5. State the Triangle Inequality.

(10) 6. A 1-ft wrench is attached to a bolt and is parallel to the ground. A force of 35 lb is applied straight upward. Give the magnitude of the torque about the center of the bolt.

$\left(1 \cdot 35 \cdot \sin\frac{\pi}{2}\right)$ lb $= 35$ lb

(10) 7. Give the equation of the plane that contains the points (-1,0,4), (-2,-3,1), and (3,-1,0).

Set $\boldsymbol{v} = \langle 1,3,3 \rangle$, $\boldsymbol{w} = \langle 5,2,\text{-}1 \rangle$, and $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle \text{-}9,16,\text{-}13 \rangle$.

$\text{-}9(x+1) + 16(y-0) - 13(z-4) = 0$

$\text{-}9x + 16y - 13z = 43$

(10) 8. Give the equation(s) (in the form of your choice) of the line that contains the points (-1,0,4) and (-2,-3,1).

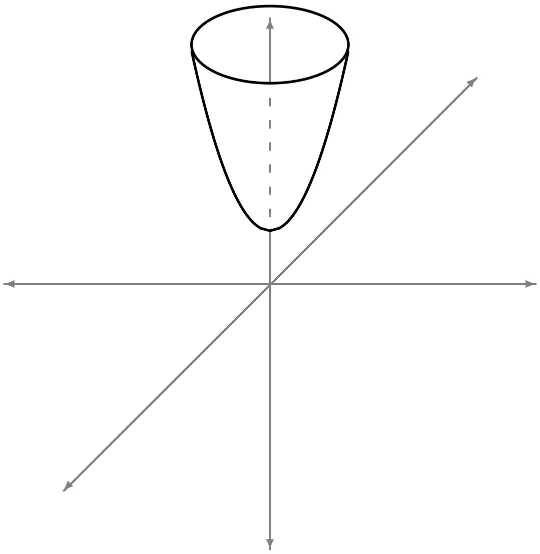
Set $\boldsymbol{v} = \langle 1,3,3 \rangle$.

$\boldsymbol{l}(t) = \langle \text{-}1,0,4 \rangle + t\langle 1,3,3 \rangle$

9. Identify and sketch each of the following.

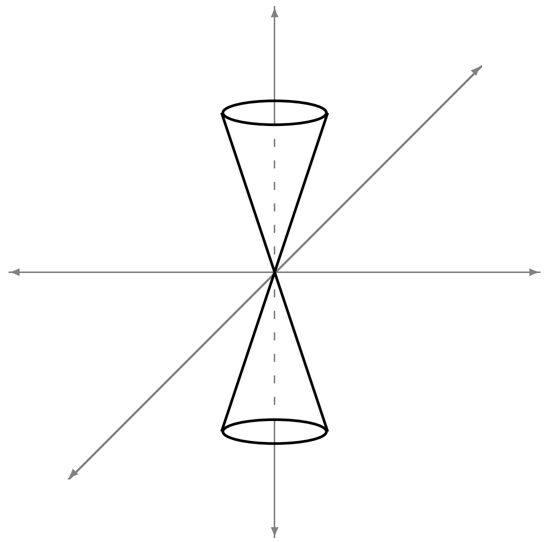
(10) a. $z = x^2 + y^2 + 2$

This is a circular paraboloid whose axis is the positive z-axis.



(10) b. $z^2 = x^2 + y^2$

This is a circular cone whose axis is the z-axis.



10. The position of an object is given by the vector-valued function $\boldsymbol{s}(t) = \langle \ln t, \sqrt{2}t, \frac{1}{2}t^2 \rangle$.

(30) a. For each of the following, give the function that describes the given property.

i. Velocity

$\boldsymbol{v}(t) = \boldsymbol{s}'(t) = \left\langle \frac{1}{t}, \sqrt{2}, t \right\rangle$

ii. Acceleration

$\boldsymbol{a}(t) = \boldsymbol{v}'(t) = \left\langle -\frac{1}{t^2}, 0, 1 \right\rangle$

iii. Speed

$$v(t) = \|\boldsymbol{v}(t)\| = \sqrt{\frac{1}{t^2} + 2 + t^2} = \sqrt{\frac{t^4+2t^2+1}{t^2}} = \sqrt{\frac{(t^2+1)^2}{t^2}} = \frac{t^2+1}{t} = t + \frac{1}{t}$$

(10) b. Find the distance traveled by the object from $t = 1$ to $t = 2$.

$$\int_1^2 \|\boldsymbol{s}'(t)\|\,dt = \int_1^2 t + \tfrac{1}{t}\,dt = \left(\tfrac{1}{2}t^2 + \ln t\right)\Big|_1^2 = 2 + \ln 2 - \left(\tfrac{1}{2} + 0\right) = \tfrac{3}{2} + \ln 2$$

11. Set $\boldsymbol{r}(t) = \langle e^t, t^3, \sin(\pi t) \rangle$.

(10) a. $\lim_{t \to 0} \boldsymbol{r}(t) = \langle 1, 0, 0 \rangle$

(10) b. $\boldsymbol{r}'(t) = \langle e^t, 3t^2, \pi\cos(\pi t) \rangle$

(10) c. Give the equation(s) (in the form of your choice) of the line that is tangent to the graph of $\boldsymbol{r}$ at the point where $t = 1$.

$\boldsymbol{r}(1) = \langle e, 1, 0 \rangle$ $\quad$ $\boldsymbol{r}'(1) = \langle e, 3, -\pi \rangle$ $\quad$ $\boldsymbol{l}(t) = \langle e, 1, 0 \rangle + t\langle e, 3, -\pi \rangle$

(10) d. $\displaystyle\int_0^1 \boldsymbol{r}(t)\,dt = \left\langle e^t, \tfrac{1}{4}t^4, -\tfrac{1}{\pi}\cos(\pi t) \right\rangle\Big|_0^1 = \left\langle e, \tfrac{1}{4}, \tfrac{1}{\pi} \right\rangle - \left\langle 1, 0, -\tfrac{1}{\pi} \right\rangle = \left\langle e - 1, \tfrac{1}{4}, \tfrac{2}{\pi} \right\rangle$

Section 7.3: Exam 2

Exam 2 Math 2673 Summer 2024 Name:________________________

12. Set $f(x,y) = \sqrt{x^2 + y^2 + 1}$.

(20) a. Give the domain and range of f.

D: $\mathbb{R}^2$

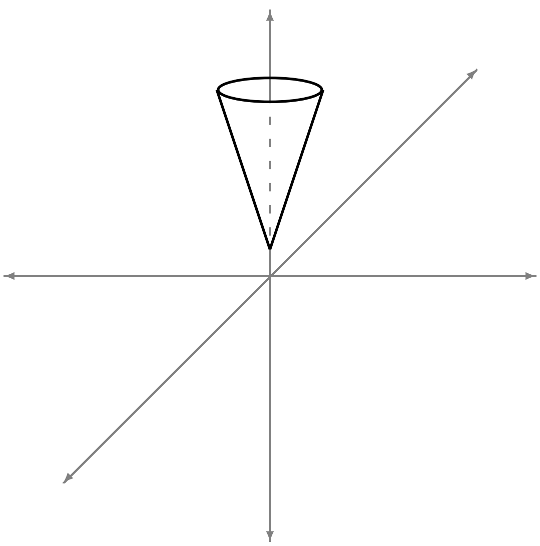
R: $[1,\infty)$

(20) b. Calculate $\lim\limits_{(x,y)\to(2,2)} f(x,y)$.

$\lim\limits_{(x,y)\to(2,2)} f(x,y) = 3$

(10) c. Sketch the graph of f.

Top half of a cone shifted up 1 unit.



(20) d. Calculate the first order partial derivatives of f.

$$f_x(x,y) = \frac{x}{\sqrt{x^2 + y^2 + 1}}$$

$$f_y(x,y) = \frac{y}{\sqrt{x^2 + y^2 + 1}}$$

(10) e. Give the equation of the plane that is tangent to the graph of f at the point (2,2,3).

$f_x(2,2) = \frac{2}{3}$

$f_y(2,2) = \frac{2}{3}$

$z - 3 = \frac{2}{3}(x-2) + \frac{2}{3}(y-2)$

$\frac{2}{3}x + \frac{2}{3}y - z + \frac{1}{3} = 0$

(10) f. Give the linear approximation of f at the point (2,2,3).

$L(x,y) = \frac{2}{3}(x-2) + \frac{2}{3}(y-2) + 3$

(10) g. Use your answer from the previous part to approximate $f(1.99,2.01)$.

$L(1.99,2.01) = \frac{2}{3}(\text{-}.01) + \frac{2}{3}(.01) + 3 = 3$

13. Calculate the following limits.

(10) a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^3 + y^3}$

DNE

$$\lim_{(x,0)\to(0,0)} \frac{xy^2}{x^3+y^3} = \lim_{x\to 0} \frac{0}{y^3} = 0$$

$$\lim_{(x,x)\to(0,0)} \frac{xy^2}{x^3+y^3} = \lim_{x\to 0} \frac{x^3}{2x^3} = \frac{1}{2}$$

(10) b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{\sqrt{x+y}}$

$$= \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{\sqrt{x+y}}$$

$$= \lim_{(x,y)\to(0,0)} \left[\sqrt{x+y}(x-y)\right]$$

$$= 0$$

14. Set $f(x,y) = x^2y - xy + \frac{1}{2}y^2$.

(30) a. Calculate all first order partial derivatives and all second order partial derivatives of f.

$f_x(x,y) = 2xy - y$

$f_y(x,y) = x^2 - x + y$

$f_{xy}(x,y) = 2x - 1$

$f_{yx}(x,y) = 2x - 1$

$f_{xx}(x,y) = 2y$

$f_{yy}(x,y) = 1$

(10) b. Find all critical points of f and apply the Second Partials Test to each one.

$f_x(x,y) = 0$

$2xy - y = 0$

$y(2x-1) = 0$

$y = 0$ or $x = \frac{1}{2}$

$f_y(x,0) = 0$

$x^2 - x = 0$

$x = 0,1$

$f_y\left(\frac{1}{2},y\right) = 0$

$\frac{1}{4} - \frac{1}{2} + y = 0$

$y = \frac{1}{4}$

CP: (0,0), (1,0), $\left(\frac{1}{2},\frac{1}{4}\right)$

$D(0,0) =$ -1

Saddle Point: (0,0,0)

$D(1,0) =$ -1

Saddle Point: (1,0,0)

$D\left(\frac{1}{2},\frac{1}{4}\right) = \frac{1}{2}$

$f_{xx}\left(\frac{1}{2},\frac{1}{4}\right) = \frac{1}{2}$

Local minimum of -$\frac{1}{32}$ at $\left(\frac{1}{2},\frac{1}{4}\right)$.

(10) c. Find the maximum and minimum values of f on the region bounded by the triangle with vertices (0,0), (1,0), and (1,1).

Recall that the maximum and minum values of f occur at either critical points or boundary points. To analyze f on the boundary of the region, consider three cases.

Case 1: $y = 0$

$f(x,0) = 0$

Case 2: $x = 1$

$f(1,y) = \frac{1}{2}y^2$

Min: $f(1,0) = 0$

Max: $f(1,1) = \frac{1}{2}$

Case 3: $y = x$

$f(x,x) = x^3 - \frac{1}{2}x^2$

Set $g(x) = x^3 - \frac{1}{2}x^2$.

$g'(x) = 3x^2 - x = x(3x-1)$

$g(0) = f(0,0) = 0$

$g(1) = f(1,1) = \frac{1}{2}$

$g\left(\frac{1}{3}\right) = f\left(\frac{1}{3},\frac{1}{3}\right) =$ -$\frac{1}{54}$

Therefore, the maximum and minimum values of f on the region are

Max: $f(1,1) = \frac{1}{2}$

Min: $f\left(\frac{1}{3},\frac{1}{3}\right) =$ -$\frac{1}{54}$

(10) 15. Set $f(x,y,z) = x^2 + y + z$. Prove that f is differentiable for all $(a,b,c) \in \mathbb{R}^3$ and find $f'_{(a,b,c)}(x,y,z)$.

***Proof*:** First, calculate

$f_x(x,y,z) = 2x$

$f_y(x,y,z) = 1$

$f_z(x,y,z) = 1$

Now consider

$$\lim_{(h,k,l)\to(0,0,0)} \frac{f(a+h, b+k, c+l) - f(a,b,c) - \left[f_x(a,b,c)\cdot h + f_y(a,b,c)\cdot k + f_z(a,b,c)\cdot l\right]}{\|(h,k,l)\|}$$

$$= \lim_{(h,k,l)\to(0,0,0)} \frac{(a+h)^2 + b + k + c + l - (a^2+b+c) - [2ah + k + l]}{\sqrt{h^2+k^2+l^2}}$$

$$= \lim_{(h,k,l)\to(0,0,0)} \frac{a^2 + 2ah + h^2 + b + k + c + l - a^2 - b - c - 2ah - k - l}{\sqrt{h^2+k^2+l^2}}$$

$$= \lim_{(h,k,l)\to(0,0,0)} \frac{h^2}{\sqrt{h^2+k^2+l^2}}$$

$$\leq \lim_{(h,k,l)\to(0,0,0)} \frac{h^2}{\sqrt{h^2}}$$

$$= \lim_{(h,k,l)\to(0,0,0)} |h|$$

$= 0$

$f'_{(a,b,c)}(x,y,z) = 2ax + y + z$

16. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x,y,z) = x \sin y^2 + x^2 e^z + yz$.

(20) a. Calculate all first order partial derivatives of f.

$f_x(x,y,z) = \sin y^2 + 2xe^z$ $\qquad$ $f_y(x,y,z) = 2xy \cos y^2 + z$ $\qquad$ $f_z(x,y,z) = x^2 e^z + y$

(10) b. Calculate $\nabla f\left(1,\sqrt{\frac{\pi}{2}},0\right)$.

$\nabla f\left(1,\sqrt{\frac{\pi}{2}},0\right) = \left\langle 3,0,1+\sqrt{\frac{\pi}{2}}\right\rangle$

(10) c. Calculate the directional derivative of f at the point $\left(1,\sqrt{\frac{\pi}{2}},0\right)$ in the direction of $\boldsymbol{u} = \left\langle \frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right\rangle$.

$\left\langle 3,0,1+\sqrt{\frac{\pi}{2}}\right\rangle \cdot \left\langle \frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right\rangle$

$= \frac{3}{\sqrt{2}} + 0 + \frac{1}{2}\left(1+\sqrt{\frac{\pi}{2}}\right)$

$= \frac{3\sqrt{2}+2+\sqrt{2\pi}}{2}$

(10) d. Give the equation for the tangent plane to the level surface $f(x,y,z) = 2$ at the point $\left(1,\sqrt{\frac{\pi}{2}},0\right)$.

$3(x-1) + \left(1+\sqrt{\frac{\pi}{2}}\right) z = 0$

(10) e. Give the equation for the normal line to the level surface $f(x,y,z) = 2$ at the point $\left(1,\sqrt{\frac{\pi}{2}},0\right)$.

$\boldsymbol{l}(t) = t\left\langle 3,0,1+\sqrt{\frac{\pi}{2}}\right\rangle + \left\langle 1,\sqrt{\frac{\pi}{2}},0\right\rangle$

(10) f. Define $x : \mathbb{R} \to \mathbb{R}$, $y : \mathbb{R} \to \mathbb{R}$, $z : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ as follows.

$x(t) = \sin t$ $\qquad$ $y(t) = e^t$ $\qquad$ $z(t) = t^3$ $\qquad$ $g(t) = f(x(t),y(t),z(t))$

Calculate $g'(0)$.

$g'(t) = f_x(x,y,z) \cdot x'(t) + f_y(x,y,z) \cdot y'(t) + f_z(x,y,z) \cdot z'(t)$

$g'(0) = f_x(0,1,0) \cdot x'(0) + f_y(0,1,0) \cdot y'(0) + f_z(0,1,0) \cdot z'(0) = (\sin 1)(1) + 0 \cdot 1 + 1 \cdot 0 = \sin 1$